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BY

Augustus

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Professor of Civil Engineering
in the Sheffield Scientific School of Yale University;
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"The Strains in Framed Structures,"
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INTRODUCTION.

CHAPTER I.

MEASUREMENT.

Physical Science.—We live in a world of matter, space and time. We do not know what these are in themselves and we cannot explain or define any one of them in terms of the others.

Thus we recognize matter in certain *states* which we call solid, **liquid** or **gaseous**. We distinguish also different *kinds* of matter, such as iron, wood, glass, water, air, etc., which we call **substances**. We also recognize limited portions of matter of definite shape and volume, such as a pebble, a rain-drop, a planet, etc., which we call **material bodies**. But what matter is in itself we do not know.

We also recognize matter as occupying space and we note successive events as occupying time. But what space and time are in themselves we do not know.

We also recognize **force** as causing change of motion of matter. But what force is in itself we do not know.

Yet although we thus know nothing of matter, space, time and force in themselves, we can and do investigate them in their *measurable relations*, and such investigation is the object of all **physical science**.

Mechanics—Kinematics and Dynamics—Statics and Kinetics.—That branch of physical science which treats of the measurable relations of space alone is called **geometry**.

That which deals with the measurable relations of *space* and *time* only, that is with pure motion, is called **kinematics** (κινήμα, motion). To the ideas of geometry it adds the idea of motion.

That which deals with the measurable relations of *space*, *time* and *matter*, involved in the study of the motion of material bodies under the action of *force*, is called **dynamics** (δύναμις, force). To the ideas of kinematics it adds the idea of force.

We divide dynamics into two parts: **statics, which treats** of material bodies at rest under the action of force, and **kinetics**, which treats of material bodies in motion under the action of force. Statics is thus a special case of dynamics which it is convenient to consider separately.

In the study of *machines*, or of moving bodies generally, under the action of force, we have to make use both of the principles of kinematics and of dynamics. The term **mechanics** is therefore used to include the general principles of both kinematics and dynamics, while their special application to machines is called **applied mechanics** or mechanism. We treat in this work of **mechanics** as thus defined, or the general principles of kinematics and dynamics.

We have then as the scheme of the present work:

$$\text{Mechanics: } \left\{ \begin{array}{l} \text{Vol. I, } \textbf{Kinematics}; \\ \left. \begin{array}{l} \text{Vol. II, } \textbf{Statics}; \\ \text{Vol. III, } \textbf{Kinetics}; \end{array} \right\} \textbf{Dynamics}. \end{array} \right.$$

Measurement.—Since then we have to do in all that follows with the measurable relations of force, matter, space and time, the subject of the measurement of these quantities should first engage our attention.

Unit.—In order to measure any quantity whatever, we must always compare its magnitude with the magnitude of another quantity *of the same kind*. The quantity thus taken as a standard of comparison is called the **unit** of measurement.

Thus the unit of length must itself be some specified length, as, for instance, one foot, one yard, one centimeter or one meter. The unit of time must be a specified time, as one second. The unit of mass must be a specified mass, as one pound or one gram or one kilogram.

The units of mass, length and time are called **fundamental units**, because not derived from any others.

Statement of a Quantity.—The complete statement of a quantity requires, therefore, a statement of the unit adopted and also a statement of the result of comparison of the magnitude of the quantity with the magnitude of the unit.

The result of this comparison is always a ratio between the magnitudes of two quantities of the same kind and is, therefore, always an abstract number.

This ratio or abstract number is called the **numeric**.

Thus we say 3 *feet*, 4 *seconds*, 5 *pounds*. In each of these cases we state both the unit and the numeric, or ratio of the magnitude of the quantity to that of the unit. Thus 3 feet denotes a quantity whose magnitude is three *times* the magnitude of one foot.

So for any quantity. In general, if L stands for any length and $[L]$ stands for the unit of length, we have $L = l[L]$, or the length equals l times the unit of length. Here l is the numeric and is an abstract number.

Again, if T is a certain interval of time, and $[T]$ stands for the unit of time, we have $T = t[T]$, or the time equals t times the unit of time. Here t is the numeric and is an abstract number.

So also if M is a certain mass and $[M]$ stands for the unit of mass, we have $M = m[M]$, or the mass equals m times the unit of mass. Here m is the numeric and is an abstract number.

Derived Unit.—A unit of one kind which is derived by reference to a unit of another kind is called a **derived unit**.

Thus the unit of area may be taken as a square whose side is one unit of length, or one square foot. The unit of volume may be taken as a cube whose edge is the unit of length, or one cubic foot. The unit of speed may be taken as one unit of length *per unit of time*, or one foot *per* second.

Such units are derived units, while the units of mass, space and time, not being thus derived from any others, are fundamental units.

Dimensions of a Derived Unit.—A statement of the mode in which the magnitude of a derived unit varies with the magnitudes of the fundamental units which compose it is a statement of the **dimensions** of the derived unit.

Thus let $[A]$ denote the unit of area and $[L]$ the unit of length. Then if $A = a[A]$ is the area of a square whose side is $L = l[L]$, where a and l are abstract numbers, we shall have $a[A] = l^2[L]^2$.

Now we shall have the numeric equation $a = l^2$, or the number of units of area equals the square of the number of units of length, provided we have $[A] = [L]^2$, or the unit of area equal to the square of the unit of length.

The statement $[A] = [L]^2$ is a statement of the dimensions of the unit of area.

Again, let $[L]$ denote the unit of length and $[T]$ denote the unit of time and $[V]$ denote the unit of speed. Then if $L = l[L]$ is any distance and $T = t[T]$ is the time occupied in describing that distance, and the mean speed is $V = v[V]$, we have

$$v[V] = \frac{l[L]}{t[T]}.$$

We shall then have the numeric equation $v = \frac{l}{t}$, or the number of units of speed is equal to the number of units of length passed over divided by the number of units of time occupied, provided we have

$$[V] = \frac{[L]}{[T]},$$

or the unit of speed equal to one unit of length *per* unit of time. This is a statement of the dimensions of the unit of speed.

Meaning of "Per."—It will be observed that the statement $[V] = \frac{[L]}{[T]}$ is read, "the unit of speed is equal to the unit of length *per* unit of time," and the word *per* is indicated by the sign for "*divided by.*"

Now we can divide the numeric l by the numeric t, because these are abstract numbers. But it would be nonsense to speak of *dividing* length by time, or a unit of length by a unit of time. We therefore avoid such a statement by the use of the word *per*. If then we give to the symbol of division this new meaning, we can then treat it by the rules which apply to the old meaning, and thus avoid the invention of a new symbol by using an old one in a new sense.

Whenever, then, the word "**per**" *is used, it can be replaced by the sign of division.*

Homogeneous Equations.—The symbols in all formulas or statements of the relations of quantities always stand for the numerics of these quantities, and the units are always understood though not written.

Thus such an equation as $v = \frac{l}{t}$ or $l = vt$ is a numeric equation, and the units are understood and must always be supplied in interpreting them. When the units are thus supplied, all the terms on both sides of the equation which are combined by addition or subtraction must always be of the same kind, whatever the system of units adopted. Such equations are called **homogeneous**.

If any numeric equation is not thus homogeneous, it is incorrectly stated.

It is also evident that all algebraic combinations of such homogeneous equations must always produce homogeneous equations. If not, some error must have been made in the algebraic work.

Error can thus often be detected in the result of an investigation without following through its successive steps, by simply inserting the omitted units, and no equation or result should be accepted, or even discussed, which does not stand this test.

Thus in the equation $l = vt$, if we supply the omitted units, we have

$$l[L] = v\frac{[L]}{[T]} \times t[T] = vt[L].$$

The equation is therefore homogeneous, since the unit of length is to be understood in both terms.

Unit of Time.—The unit of time ordinarily adopted in dynamics is the **second** or some multiple of the second.

It is the time of vibration of an isochronous pendulum which vibrates or beats 86400 times in a mean solar day of 24 hours, each hour containing 60 minutes and each minute 60 seconds ($24 \times 60 \times 60 = 86400$).

The sidereal day contains 86164.09 of these mean solar seconds.

Unit of Length.— The unit of length ordinarily adopted in dynamics is the **foot** or the **meter** or some multiple of these.

Unit of Mass.—The unit of matter or mass ordinarily adopted in dynamics is the **pound** or the **kilogram.**

Standard Unit.—All units adopted are defined by reference to certain **standard** units. A standard unit, in general, should possess, so far as possible, a permanent magnitude unchanged by lapse of time and unaffected by the action of the elements or by change of place or temperature. It should be capable of exact duplication and should admit of direct and accurate comparison with other quantities of the same kind.

Standard Unit of Time.—The *standard* unit of time is the period of the earth's rotation, or the *sidereal day.* This has been proved by Laplace, from the records of celestial phenomena, not to have changed by so much as one eight-millionth part of its length in the course of the last two thousand years.

The length of the solar day is variable, but the mean solar day, which is the exact mean of all its different lengths, is the period already mentioned, which furnishes the second of time. It is 1.00273791 of a sidereal day.

The second can therefore be defined, with reference to the standard unit of time, as the time of one swing of a pendulum so adjusted as to make 86400 oscillations in 1.00273791 of a sidereal day.

Standard Units of Length.—The English standard unit of length is the length of a standard bronze bar, deposited in the Standards Department of the Board of Trade in London.

Since such a bar changes in length with its temperature, the length is taken at the specified temperature of 62° Fah.

The length of this bar at this temperature is the English *standard unit of length,* and is called the **standard yard.** Accurate copies of this standard are distributed in various places, and from these all local standards of length are derived.*

The foot is defined as one third the length of the standard yard at 62° Fah.

---

* The English standard yard is 1 part in 17230 shorter than the U. S. copy.

The French standard of length is the *meter*, and is the length of a bar of platinum at the temperature of melting ice, or 0° C. This bar is preserved at Paris. Its length was intended to be the ten-millionth part of a quadrant of the earth's meridian through Paris.

The quadrant of the meridian through Paris is 10001472 standard meters, according to *Colonel Clarke's* determinations of the size and figure of the earth, which are at present the most authoritative, and thus the standard Paris meter is slightly less than the length upon which it was founded. The *material bar* is therefore the standard, just as is the case with the English standard.

The relation of the meter to the meridan was intended as a means of reproduction in case of destruction of the standard, but in such case the standard would probably be reproduced from the best existing copies.

This was actually the case with the original English standard, which was destroyed by fire in 1834. It had been originally defined as having at 62° Fah. a length of $\frac{36}{39.1393}$ of the length of a pendulum vibrating seconds in the latitude of London at the sea-level. But this provision for its restoration was repealed and a new standard bar was constructed from authentic copies of the old one.

The English inch, or the 36th part of the length of the standard yard, is very nearly equal to the five-hundred-millionth part of the length of the earth's polar axis $\left(\frac{1}{500482296}\right)$

The utility of the standard, however, does not depend upon any such earth relations, the only value of which is for reproduction in case of destruction—a value which, as we have seen, is practically disregarded.

The ultimate standards are therefore the *actual bars*.

Standard Units of Mass.—The English standard unit of mass is a piece of platinum deposited in the Office of the Exchequer at London and called the "Imperial Standard Pound Avoirdupois."

The French standard unit of mass is a piece of platinum preserved at Paris and called the kilogram.

Unit of Angle.—There are two units of angle in use, the **degree** and the **radian**.

The degree is that angle subtended at the centre of any circle by an arc equal in length to $\frac{1}{360}$ part of the circumference of that circle. It is subdivided sexagesimally into degrees (°), minutes (′), and seconds (″). The seconds are subdivided decimally. Minutes and seconds of *time* are distinguished by being written *min.*, *sec.*

The radian is that angle subtended at the centre of any circle by an arc equal in length to the radius. It is subdivided decimally.

If then the length of any arc is $s[L]$, or s units of length, and the length of the radius is $r[L]$, or r units of length, and if the angle subtended at the centre is θ radians, we have

$$\theta = \frac{s[L]}{r[L]} = \frac{s}{r}, \quad \text{or} \quad r\theta = s.$$

The number of radians in any angle is then found by dividing the number of units of length in the subtending arc by the number of units of length in the radius, and this number is *independent of the particular unit of length adopted*, whether feet or centimeters.

If the subtending arc is the entire circumference, the number of radians is $\frac{2\pi r}{r} = 2\pi$. Hence 2π radians correspond to 360 degrees, or 1 radian corresponds to $\frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = 57.29578$ degrees $= 57^\circ\ 17'\ 44''.8$.

Any angle expressed in radians may then be converted into degrees by multiplying the number of radians by $\frac{180^\circ}{\pi} = 57.29578$ degrees $= 1$ radian.

Any angle expressed in degrees may be converted into radians by multiplying the number of degrees by $\frac{\pi}{180} = 0.0174533$ radians $= 1$ degree.

(1) *Express* 12° 34′ 56″ *in terms of radians; and* 3 *radians in terms of degrees.*

Ans. 0.2196 radians; 171° 53′ 14″.424.

(2) *The radius of a circle is* 10 *feet; what is the angle subtended at the center by an arc of* 3 *feet?*

Ans. $\frac{3}{10}$ radian, or 17° 11′ 19.″44.

(3) *How much must a rail* 30 *feet long be bent in order to fit into a curve of half a mile radius?*

Ans. $\frac{1}{88}$ radian, or 0° 39′ 3″.92.

(4) *Express* 45 *degrees in terms of radians, and* 4.5 *radians in terms of degrees.*

Ans. $\frac{\pi}{4}$ radians $= 0.7854$ radians; 257° 49′ 51″.636.

(5) *The angle subtended at the centre of a circle by an arc whose length is* 1.57 *feet is* 15°; *what is the radius?*

Ans. $\frac{s}{r} = \frac{15\pi}{180}$, or $r = \frac{1.57 \times 180}{15\pi} = 6$ ft.

(6) *What is the sin* $\frac{\pi}{6}$ *radians; cos* $\frac{\pi}{6}$ *radians; cos* $\frac{\pi}{3}$ *radians; tan* $\frac{\pi}{4}$ *radians?*

Ans. 0.5; $\frac{1}{2}\sqrt{3}$; 0.5; 1.

(7) *Express in degrees and in radians the angle made by the hands of a clock at* 35 *minutes past* 3 *o'clock.*

Ans. 102.5 degrees; 1.79 radians.

Unit of Conical Angle.—Let the area of any portion of the surface of a sphere be $A[A]$, or A units of area, and let the square of the radius be $r^2[A]$, or r^2 units of area.

If lines are drawn from the centre C of the sphere to every point of the area, they form a cone, and the angle subtended at the centre C by the area we call a **conical angle.**

The conical angle subtended at the centre of a sphere by a portion of its surface whose area is equal to the square of its radius we call a **square radian.** If we denote the conical angle subtended by the area A by ω square radians, we have

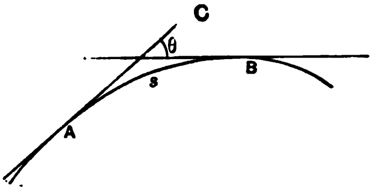
$$\omega = \frac{A[A]}{r^2[A]} = \frac{A}{r^2}, \quad \text{or} \quad r^2\omega = A.$$

The number of square radians in any conical angle is thus found by dividing the number of units of area in the subtending area by the number of units of area in the square of the radius, and this number *is independent of the unit of area adopted.*

If the subtending area is the entire surface of the sphere, the number of square radians is $\frac{4\pi r^2}{r^2} = 4\pi$. Hence the surface of a sphere subtends a conical angle of 4π square radians.

[The terms *solid angle* and *solid radian* are usually employed in place of conical angle and square radian as defined, but as they seem in no way descriptive, we have employed the latter terms as more expressive.]

Curvature.—The direction of a plane curve at any point is that of the tangent to the curve at this point.



Thus the direction of the curve AB at the point A is that of the tangent AC.

The *change* of direction between any two points of a plane curve is the angle between the tangents at these two points, and is called the **integral curvature.**

Thus the angle θ, or change of direction between the tangents at A and B, is the integral curvature for the curve between A and B.

The integral curvature for any portion of a plane curve divided by the length of that portion is the *mean curvature.*

Thus if the length from A to B of the curve is $s[L]$, or s units of length, the mean curvature is $\frac{\theta[\theta]}{s[L]}$. Since θ is given in radians, the unit of curvature is *one radian per unit of length of arc.* When we say, therefore, that the mean curvature is $\frac{\theta}{s}$, we mean $\frac{\theta}{s}$ radians per unit of length of arc.

The limiting value of the mean curvature when the two points are indefinitely near is called the *curvature.*

The curvature, therefore, is the *limiting rate of change of direction per unit of length of arc.* Its unit is one radian per unit of length of arc.

Curvature of a Circle.—If the curve is a circle, the angle at the centre between the radii at A and B will be equal to the angle θ between the tangents at A and B.

We have then $\theta = \frac{s}{r}$ radians. The mean curvature is then $\frac{\theta}{s} = \frac{1}{r}$ radians per unit of length of arc.

Since this is independent of s, the curvature at every point of a circle is constant and equal to the mean curvature for any two points, viz., $\frac{1}{r}$ radians per unit of length of arc.

Curvature of any Plane Curve.—For any plane curve whatever a circle can always be described whose curvature is the same as that of the given curve at the given point. This is the *circle of curvature* of the curve at that point. Its radius is the *radius of curvature* of the curve at that point.

If then ρ is the radius of curvature of a curve at any given point, the curvature of that point is $\frac{1}{\rho}$ radians per unit of length of arc.

Since curvature then depends only upon the radius of curvature, the circle is the only curve whose curvature is constant.

(1) *A circle has a radius of* 10 *feet. What is its curvature?*

Ans. $\frac{1}{10}$ radian per foot of arc, or 5°.73 per foot of arc.

(2) *If the radius is* 10 *yards, what is the curvature?*

Ans. $\frac{1}{10}$ radian per yard of arc, or 5°.73 per yard of arc, or $\frac{1}{30}$ radian per foot of arc, or 1°.91 per foot of arc.

Dimensions of Unit of Curvature.—If C is the curvature and c the number of units of curvature, we have by definition $c[C] = \frac{\theta[\theta]}{s[L]}$, where $[C]$ is the unit of curvature, and $[\theta]$ is the unit of angle, $[L]$ the unit of length, and θ, s the number of units of angle and length.

We shall always have $c = \frac{\theta}{s}$, provided we take $[C] = \frac{[\theta]}{[L]}$, that is, provided the unit of curvature is equal to the unit of angle divided by the unit of length.

This is a statement of the dimensions of the unit of curvature.

The unit of curvature is then one unit of angle per unit of length of arc, as, for instance, one radian per foot of arc, or one degree per foot of arc.

A railway curve has a length of one mile, the curvature is uniform, and the integral curvature is 30 *degrees. What is the curve, the curvature, and the radius of curvature?*

Ans. A circle; 0.5236 radian per mile arc; 1.9 miles radius.

Tables of Measures.—We shall deal in the course of this work with many other derived units, which will be explained as they occur. It will be useful to collect here for convenience of reference a number of such units.

I. MEASURES OF SPACE.

A. LENGTH.

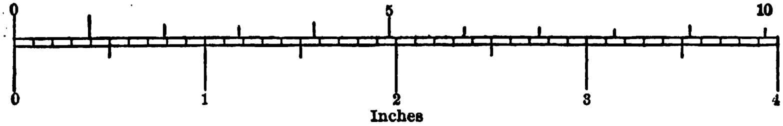


TABLE 1.

1 centimeter = 0.3937079 inch

1 meter = { 39.37079 inches, 3.2809 feet }

1 kilometer = { 0.62137 mile, 0.535987 nautical m. }

TABLE 2.

1 inch = 2.539954 centimeters

1 foot = 30.479449 "

1 yard = 0.91438347 meter

1 mile = 1.60935 kilometers

1 nautical mile, or 6080.26 ft. } = 1.85327 kilometers

The following are approximate :

The centimeter is about ⅜ inch. The meter is about 3 ft. 3⅜ inches.
The decimeter is about 4 inches. One kilometer is about ⅝ of a mile.
Distance from pole to equator is about 10000000 meters.
Earth's polar radius is about 500000000 inches.

B. AREA.

TABLE 3.

1 sq. centimeter	= 0.155006 sq. inch
1 sq. meter	= 10.7643 sq. feet
1 sq. hectometer, or 1 hectare	= 2.47114 acres
1 sq. kilometer	= 0.38611 sq. mile

TABLE 4.

1 sq. inch	= 6.45137 centimeters
1 sq. foot	= 928.997 "
1 sq. yard	= 0.836097 meter
1 acre	= 0.404672 hectare
1 sq. mile	= 2.58989 sq. kilometers

C. VOLUME.

TABLE 5.

1 cubic centimeter	= 0.0610271 cu. inch
1 liter, or 1 cubic decimeter	= 61.0271 cu. inches = 1.76172 pints
1 cubic meter	= 35.3166 cubic feet

TABLE 6.

1 cubic inch	= 16.3866 cubic centimeters
1 cubic foot	= 28.3153 liters
1 cubic yard	= 0.764513 cubic meter
1 pint	= 0.567627 liter
1 gallon	= 4.54102 liters

II. MEASURES OF MASS.

TABLE 7.

1 centigram	= 0.154323 grain
1 gram	= 15.4323 grains 0.0353739 oz.
1 kilogram	= 2.20462 lbs.

TABLE 8.

1 grain	= 0.064799 gram
1 oz.	= 28.3496 grams
1 lb. or 16 oz.	= 0.453593 kilogram
1 ton or 2240 lbs.	= 1016.05 kilos

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1 gram = mass of 1 cubic centimeter of pure water at 4° C.

1 kilogram = 1 liter of pure water at 4° C.

1 gallon = 277.274 cubic inches. The gallon contains 10 lbs. of pure water at 62° F.

1 cubic foot of water contains about 1000 oz. or 62½ lbs.

The pint contains 20 fluid oz.

Acceleration of gravity at London = 32.182 feet-per-second per second = 980.889 centimeters-per-second per second. Average value 32⅙ feet-per-second per second or 980.3 centimeters-per-second per second.

1 dyne = force which will give a mass of 1 gram an acceleration of 1 centimeter-per-second per second = about $\frac{1}{981}$ weight of gram = weight of about 1 milligram.

1 poundal = force which will give a mass of 1 pound an acceleration of 1 foot-per-second per second = about weight of ½ oz.

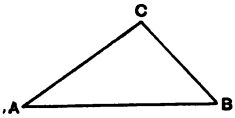
CHAPTER II.

POSITION. TERMS AND DEFINITIONS.

Point.—A mathematical point has neither length, breadth, nor thickness. It is therefore without dimensions and indicates position only.

Point of Reference.—When we speak of a point as having position, some other point or points must always be assumed, by reference to which the position is given. Such a point is a **point of reference.** It is also called a **pole, or origin.**

Position then is always relative. We know nothing of "absolute" position.



Thus the position of the point *C* is known with respect to *A* when we know the length of the line *AC* and the angle *BAC* or the direction of the line *AC*. The points *A* and *B* are points of reference, by means of which *C* is located.

Position of a Point.—The position of a point with reference to other assumed points is then known when we have sufficient data to locate it. These data give rise to two methods of location :

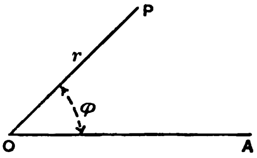
1st, by polar co-ordinates.

2d, by Cartesian co-ordinates, so called because first employed by Descartes.

Plane Polar Co-ordinates.—The data necessary for locating a point by polar co-ordinates, when the point is situated in a given plane, consist of a distance and an angle. If the point is not in a known plane, of a distance and two angles.

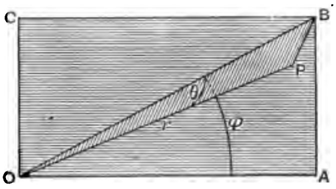
Thus, if the point *P*, in the plane of this page, is to be located, we first assume a line *OA* in the plane, as a line of reference. Then the position of *P* with reference to *O* is given by the angle *AOP* and by the distance *OP*.

The assumed point *O* is called the pole: *OA* is the **line of reference**; the distance *OP* is called the **radius vector**, and its magnitude is usually denoted by *r* ; the angle *AOP* is the **direction angle**; its magnitude is denoted by ϕ, and it is measured around from *OA* to the left.



The polar co-ordinates for a point in a given plane are therefore r and ϕ, or a distance and an angle. These are *plane* polar co-ordinates.

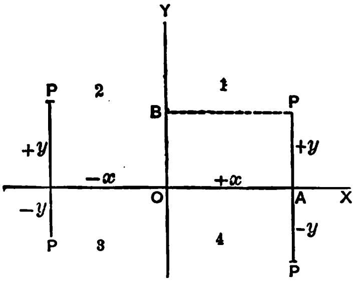
Space Polar Co-ordinates.—If the point *P* is not in a given plane, we assume as before a pole *O*, and a reference line *OA* in space. Through this line we assume any plane, as the plane of this page, *OABC*, and let *OB* be the intersection of this plane with a plane *OPB*, perpendicular to it and passing through *OP*. The location of *P* is then given by the length *OP* or the radius vector *r*, the angle *AOB* or ϕ, and the angle *BOP* or θ.



The polar co-ordinates for a point not in a given plane are therefore r, ϕ and θ, or a distance and two angles. These are *space* polar co-ordinates.

If *O* is a point on the earth's surface, and the reference line *OA* is a north and south line in the plane of the horizon, the angles θ and ϕ would be the astronomical azimuth and altitude of the point *P*.

Cartesian Co-ordinates.—Plane.—The data necessary for locating a point by Cartesian co-ordinates, if the point is in a known plane, consists of two distances, parallel to two assumed lines of reference in that plane, passing through the point of reference, which is called **the origin**. The assumed lines of reference are usually taken at right angles.



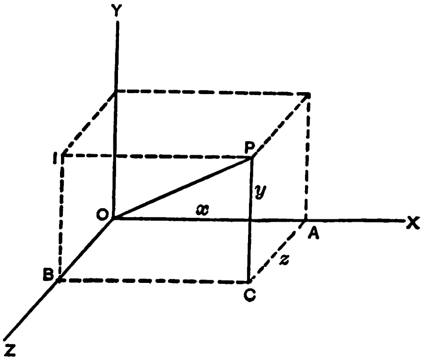
Thus if the point *P* is known to be in the plane of this page, we assume any origin *O* and draw two reference lines *OX* and *OY* through *O* in this plane and at right angles. These two lines are called the **axes of co-ordinates**, the horizontal one the *axis of x*, or the *x* axis, the other the *axis of y*, or the *y* axis. The distance *BP* or *OA* is denoted by x and called the **abscissa** of the point *P*. The distance *AP* is denoted by y and called the **ordinate** of the point *P*.

The abscissa x is positive to the right, negative to the left of the origin, while the ordinate y is positive when laid off above and negative when below the origin.

Any point in the plane is thus located with respect to *O*. If a point is in the first quadrant, its co-ordinates are $+x$, $+y$; if in the second quadrant, $-x$ and $+y$; if in the third quadrant, $-x$ and $-y$; if in the fourth quadrant, $+x$ and $-y$.

When the point is in a known plane, the co-ordinates are called *plane* co-ordinates.

Space Co-ordinates. — If the point *P* is not in a known plane, we take three axes through the origin, all at right angles usually. Two of these we may denote by *X* and *Y* as before; the third, at right angles to the plane of *XY*, we call the axis of z, or the z axis.



Thus the position of the point P is given by the distance $OA = x$, the distance $AC = z$, and the distance $CP = y$.

These are the *space co-ordinates* of the point P.

The signs prefixed to the co-ordinates indicate the quadrant in which the point is located as before. Thus, $+x$, $+y$ and $\pm z$ denote a point in the first quadrant either in front of or behind the plane of XY; $-x$, $+y$ and $\pm z$, a point in the second quadrant, either in front of or behind the plane of XY; $-x$, $-y$, $\pm z$, and $+x$, $-y$, $\pm z$, points in the third and fourth quadrants, either in front of or behind the plane of XY.

Direction Cosines.—If we join the origin O and the point P by a line, and denote the angle of OP with the x axis by α, with the y axis by β, and with the z axis by γ, we have the relations

$$x = OP \cos \alpha, \quad y = OP \cos \beta, \quad z = OP \cos \gamma.$$

These cosines are called the **direction cosines** of OP.

Since OP is the diagonal of a parallelogram, we have

$$OP^2 = x^2 + y^2 + z^2 = OP^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

Hence

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, therefore, any two of these direction cosines are given, the third can always be fonnd.

Since $\cos 2\alpha = 2 \cos^2 \alpha - 1$, $\cos 2\beta = 2 \cos^2 \beta - 1$, $\cos 2\gamma = 2 \cos^2 \gamma - 1$, we have also

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

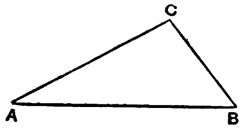
Again, since $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$, we have also

$$\cos(\alpha + \beta) \cos(\alpha - \beta) + \cos^2 \gamma = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Dimensions of Space.—A point in a given line is at once located by a statement of the distance of the point from either end of the line.

A point in a given plane is located either by two distances or by a distance and an angle.

A point in space is located either by three distances or by a distance and two angles.



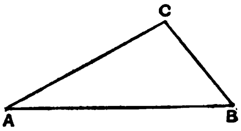
Hence space is said to have three dimensions, a plane surface to have two dimensions, and a line one dimension.

A point has no dimensions and indicates position only.

System.—Any definite and limited assemblage of points is called a **system**. Thus the assemblage of points represented by A, B, C, constitutes a system.

Configuration.—The relative position or arrangement of any system or assemblage of points is called the **configuration** of the system.

A knowledge of the configuration of a system at any instant requires a knowledge of the relative position of every point of the system with reference to every other point at that instant.



Thus the configuration at any instant of the system represented by the points A, B, C, is known when the position at that instant of A relative to B and C, of B relative to A and C, and of C relative to A and B, are known; that is, when all the sides and angles of the triangle are known at that instant.

Rigid System.—When the configuration does not change, the system is a **rigid system.**

Thus when the angles and sides in the triangle *ABC* remain unchanged, the system is rigid.

Rest.—When the straight lines drawn from a point to any assumed points of reference do not change their length or inclination to each other, the point is at **rest** with reference to these points.

The points of a rigid system are therefore at rest relatively to each other. The entire system may, however, be in motion with reference to some external point.

Rest, then, like position, is relative only. We know nothing of "absolute" rest.

Thus if the lines *CA* and *CB* in the preceding figure do not change in length and the angle *ACB* does not change, the point *C* is at rest with reference to *A* and *B*.

The system is then a rigid system, and *A* is at rest with reference to *C* and *B*, and *B* is at rest with reference to *A* and *C*.

The entire system of *A*, *B*, *C* may, however, be in motion with reference to some point external to the system.

The stationary objects in a room are all at rest relatively to each other. The straight lines joining any three do not change their length or inclination to each other.

But, as we know, all these objects partake of the motion of the earth, and are therefore not at rest with reference to the sun.

Motion.—Motion is change of position. A point **moves** when the straight lines joining it to the points of reference change either in length or inclination to each other.

Motion, therefore, like rest and position, is always relative. Such terms as "absolute" position, "absolute" rest, and "absolute" motion have no scientific value. All our knowledge, both of position and change of position, must be essentially relative.

"When a man has acquired the habit of putting words together, without troubling himself to form the thoughts which ought to correspond to them, it is easy for him to frame an antithesis between this relative knowledge and a so-called absolute knowledge, and to point out our ignorance of the absolute position of a point as an instance of the limitation of our faculties. Any one, however, who will try to imagine the state of a mind conscious of knowing the absolute position of a point will ever after be content with our relative knowledge." ("*Matter and Motion*," by J. Clerk Maxwell. Pott, Young & Co., New York, 1876.)

Path of a Point.—The line joining the successive positions of a point during its motion is called its **path.**

The path thus described by a point may be either a straight or a curved line, or a combination of straight and curved lines. If the path is without angles or abrupt changes of direction, it is **continuous.**

Motion of Translation.—When a rigid system moves so that every straight line in it joining every two points remains always parallel to itself, the system is said to have a motion of **translation.**

The paths of all the points are therefore parallel at every instant and equal for any given interval of time, and the translation of the system is that of any one of its points.

Motion of Rotation.—When a rigid system moves so that all its points describe arcs of circles in parallel planes about a common straight line or **axis** passing through the centers of the circles and perpendicular to their planes, the system is said to **rotate** or have a motion of rotation about that axis.

Since the system is rigid, every point must describe an equal angle in the same time.

A point has no dimensions and therefore cannot have motion of rotation, but only one of translation.

Combined Translation and Rotation.—A rigid system may have a motion of rotation and translation at the same time. Thus, for instance, a rolling wheel has a motion of rotation about an axis through the hub at right angles to the plane of the wheel, while at the same time every point of the wheel has a motion of translation.

Material Point or Particle.—We have just seen that a point cannot have motion of rotation. Rotation is possible only to systems of points.

A material body so small that the distances between its points may be neglected is called a **particle**.

When the body is not small, whatever its magnitude, if the distances between its various points *have no influence upon the motion considered*, we call the body a **material point** or **particle** and may represent it by a point without dimensions.

Thus if we are investigating the motion of the earth about the sun, so far as the motion of translation of the earth is concerned, we may regard both the earth and sun as points. But we cannot treat them as points when we wish to study their rotation.

Material System.—A number of material points or particles constitute a **material system**. When we confine our attention to such a system, all relations or actions between one point of such a system and another are called *internal* relations or actions. Those between the whole or any part of the system and bodies not included in the system are called *external* relations or actions.

KINEMATICS.

GENERAL PRINCIPLES.

CHAPTER I.

SPEED.

Kinematics.—That branch of science which treats of the measurable relations of time and space only, that is, of pure motion, is called **kinematics**. It adds to the ideas of pure geometry the idea of motion.

Mean Speed of a Point.—The distance described by a moving point per unit of time is called the **mean speed** of the point. Therefore the number of units of distance described in a given time, divided by the number of units in that time, gives the number of units of mean linear speed. The mean speed is then the mean time-rate of motion in the path.

When the mean speed varies with the interval of time it is **variable**. When it has the same magnitude no matter what the interval of time it is **uniform**. A point moving with uniform mean speed evidently describes equal distances in equal times.

Instantaneous Speed of a Point.—The limiting value of the mean speed when the interval of time is indefinitely small is called the **instantaneous speed**.

When the instantaneous speed at any instant is equal to the mean speed for any interval of time it is **uniform**. When the instantaneous speed is **variable** the mean speed has different values for equal intervals of time.

The term speed always signifies instantaneous speed unless otherwise specified.

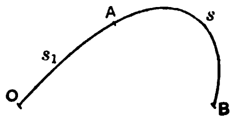
Dimensions of the Unit of Speed.—Let us denote any speed by V, the unit of speed by $[V]$ and the number of units of speed by v, so that $V = v[V]$. Then if $[L]$ is the unit of length and s the number of units of length, or the distance passed over in the time $t[T]$, we have by definition

$$v[V] = \frac{s[L]}{t[T]}.$$

We shall always have the numeric equation $v = \frac{s}{t}$, if we take

$[V] = \frac{[L]}{[T]}$, or the unit of speed equals the unit of length per unit of time.

This is the statement of the dimensions of the unit of speed. The unit of speed is therefore always taken as one unit of length per unit of time, as, for instance, one foot per second.



Numeric Equations of Speed.—If s_1 denotes, then, the number of units in the initial distance OA, *measured along the path*, of a moving point from a fixed point O in the path, taken as origin, and s the number of units in the final distance OB, *measured along the path*, from the same origin O, and t the number of units in the interval of time in describing the distance $AB = s - s_1$, we have for the mean speed the numeric equation

$$v = \frac{s-s_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the interval of time is indefinitely small we have, in the notation of the Calculus, dt in place of t, and ds in place of $s - s_1$. The instantaneous speed, or the *speed*, is then

$$v = \frac{ds}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Sign of Speed.—From (1) we see that if OB or s is numerically greater than OA or s_1, the value of v will be positive, or v equals a plus (+) quantity. If, however, s_1 is numerically greater than s, the value of v will equal a negative (—) quantity. When, then, the distance from the origin, measured along the path, is increasing, the value of v is positive (+). When the distance is decreasing, the value of v is always negative (—). Moreover, if s_1 is on the opposite side of the origin from s, we have $v = \frac{s + s_1}{t}$.

Equation (1) will therefore *hold good generally* if we take distances from the origin in one direction as (+) and in the other direction as (—). In such case, if the value of v comes out (+) it indicates motion in the assumed (+) direction, and if (—) it indicates motion in the other direction. If t comes out (—) it denotes time before the start, if (+) time after the start or beginning of motion.

Speed a Scalar Quantity.—It will be evident from the preceding that the sign of v has no reference to any special direction in space. It simply indicates that the distance along the path from the origin is increasing or decreasing, without reference to the actual direction of the path at any instant.

Speed, then, whether uniform or variable, mean or instantaneous, is independent of direction of the path. A point moving with any given speed has that speed no matter what the shape of the path.

Speed, therefore, is a quantity which has magnitude and sign, but is independent of direction. Such a quantity is called a **scalar** quantity.

The student is cautioned here not to confound speed with "velocity," which, as we shall see hereafter (page 43), has direction as well as sign and magnitude. Such a directed quantity is called a **vector** quantity.

Homogeneous Equations.—We have already called attention in the Introduction (page 3) to the fact that the units in all numeric equations are always understood, and when these units are inserted the equation must be **homogeneous**, that is, every term in it must stand for a quantity of the same kind. When this is not the case some error must have been made in the derivation of the equation, and the relations indicated by it are incorrectly stated. By simply inserting the units, then, in each term of any numeric equation we can at once check the result arrived at and often discover without further investigation if the result is incorrect.

Thus suppose that the result of some investigation is expressed by

$$3s + 2t = 10v.$$

Without reference to the various steps by which this result may have been reached, we can at once say that the result is incorrect.

Thus if we insert the units, we have

$$3s[L] + 2t[T] = 10\frac{[L]}{[T]},$$

and we see at once that the quantities in each term are not the same. The equation is not homogeneous. If, however, we had

$$3s + 2tv = 10vt,$$

this equation is homogeneous, because when we insert the units we have

$$3s[L] + 2t[T]v\frac{[L]}{[T]} = 10v\frac{[L]}{[T]}t[T], \quad \text{or} \quad 3s[L] + 2tv[L] = 10vt[L].$$

Here all the terms are quantities of the same kind, and the equation is homogeneous. The relation expressed by it is possible, that expressed by the first is impossible, because we cannot add and subtract quantities of different kinds. It does not follow that the relation $3s + 2tv = 10vt$ is correct. It may still have been incorrectly deduced. All we can say is it is not on its face absurd, while $3s + 2t = 10v$ is manifestly so.

The student should make it a rule to first test in this manner any equation the truth of which is suspected, as it may often save him the trouble of examining in detail the entire investigation by which it has been deduced. If, however, it stands this test, then the derivation must be examined also.

EXAMPLES.

(1) *Water issues from an orifice having an area of cross-section denoted by a, with a speed of v. If the discharge in cubic feet is denoted by q, criticise the formula q = av.*

Ans. Since a is the area of cross-section, its unit must be the unit of area, as, for instance, one square foot. The unit of v is the unit of speed, or one foot per second. The unit of q *by statement* must be the unit of volume, or one cubic foot. We have then

$$q \text{ cu. ft.} = a \text{ sq. ft.} \times \frac{v \text{ ft.}}{1 \text{ sec.}} = av\,\frac{\text{cu. ft.}}{1 \text{ sec.}}$$

The equation is therefore not homogeneous. We have cubic feet on one side, equal to cubic feet per second on the other. If, however, q denotes discharge in cubic feet *per second* instead of discharge, then the equation becomes homogeneous.

Even then it does not follow that the equation is correct as compared with fact. The actual discharge per second may be less than given by $q = av$. But the equation as corrected is homogeneous and *may* be correct, whereas before we *know* it is incorrect, because it states an impossible equality between unlike quantities.

(2) *A passenger sitting in a railroad car counts n rails passed over in 20 seconds. If the length of a rail is 29⅓ feet, what is the speed in miles per hour?*

Ans. n miles per hour. The number of rails in 20 seconds is the number of miles per hour.

(3) *If the radius of the earth is 4000 miles, and if it describes a path of 600 millions of miles in 365¼ dags, find (a) the speed of a point on the equator with reference to a fixed point at the equator, disregarding the motion about the sun; (b) the speed in the path with reference to a fixed point in the path, disregarding the motion of rotation.*

Ans. (a) 1535.89 ft. per sec.; (b) 19.01 miles per sec. nearly.

(4) *If the unit of speed is taken at 30 feet per second, and the unit of length at 20 inches, what should be the corresponding unit of time?*

Ans. $[V] = \frac{[L]}{[T]}$ or the unit of speed must always equal the unit of length per unit of time. Hence $[T] = \frac{[L]}{[V]} = \frac{20 \text{ in.}}{\frac{30 \text{ ft.}}{1 \text{ sec.}}}$ or $[T] = \frac{20 \text{ in.} \times 1 \text{ sec.}}{30 \times 12 \text{ in.}} = \frac{1}{18}$ sec.

(5) *If 1 minute is the unit of time adopted, and 1 decimeter per second is the unit of speed, what is the unit of length?*

Ans. $[L] = [V] \times [T] = \frac{1 \text{ dec.}}{1 \text{ sec.}} \times 60 \text{ sec.} = 60$ decimeters.

(6) *The distance of a moving point from a fixed point measured in its path is given by $s = at + bt^2$, where s is the number of feet passed over in the number of seconds t. (a) What is the unit of a and b? (b) What is the mean speed between the beginning of the 6th and the end of the 12th second from the start? (c) What is the instantaneous speed?*

Ans. (a) In order that the equation may be homogeneous, a should be given in ft. per sec. units and b in ft.-per-sec. per sec. units. (b) For $t = 5$ sec. the space passed over is $5a + 25b$. For $t = 12$ it is $12a + 144b$. The distance passed over in the interval $12 - 5 = 7$ sec. is $7a + 119b$. Hence the mean speed is $\frac{7a + 119b}{7} = a + 17b$. ($c$) We have for $t = t_1$ $s_1 = at_1 + bt_1^2$ and hence $s - s_1 = a(t - t_1) + b(t^2 - t_1^2)$ or $\frac{s - s_1}{t - t_1} = a + b(t + t_1)$. When the interval of time $t - t_1$ is indefinitely small the instantaneous speed is $a + 2bt$.

By Calculus $\frac{ds}{dt} = a + 2bt$

(7) *The distance of a moving point from a given point in its path is given by $s = 3 + 8t$, where s is the number of feet passed over in the number of seconds t. (a) What is the unit for 3 and 8? (b) What is the mean speed? (c) What is the instantaneous speed?*

Ans. (a) The number 3 should stand for 3 ft., the number 8 for 8 ft. per sec. (b) When $t = 0$, the initial distance is $s_1 = 3$ ft. Therefore $\frac{s - s_1}{t} = 8$ ft. per sec. (c) Since this is constant, the instantaneous speed is the same. Or we can write $\frac{s - s_1}{t - t_1} = 8$ ft. per sec. which we see is independent of the interval $t - t_1$. Or $\frac{ds}{dt} = 8$ ft. per sec.

(8) *Suppose the distance is given by* $s = 7t + 8t^2$. (*a*) *What is the mean speed and the instantaneous speed?* (*b*) *What is the mean speed between the beginning of the* 10*th and the end of the* 12*th second?* (*c*) *What is the instantaneous speed at the end of the* 6*th second?*

Ans. (*a*) $\frac{s - s_1}{t - t_1} = 7 + 8(t + t_1) =$ mean speed. $7 + 16t =$ instantaneous speed. (*b*) 175 ft. per sec. (*c*) 103 ft. per sec.

(9) *A train runs* 40 *miles per hour for half an hour,* 30 *miles an hour for* 20 *minutes, and* 36 *miles an hour for* 40 *minutes. Find its mean speed.*

Ans. 36 miles per hour.

(10) *A point moves in a circle whose radius is* 25 *feet and makes* 6 *revolutions in* 15.708 *seconds. What is the mean speed?*

Ans. 60 ft. per sec.

(11) *Reduce* (*a*) 60 *feet per minute to centimeters per second;* (*b*) 1 *kilometer per hour to centimeters per second;* (*c*) 36 *feet per second to yards per minute;* (*d*) 10 *yards per second to kilometers per hour.*

Ans. (*a*) $\frac{60 \text{ ft.}}{1 \text{ min.}} = \frac{60 \times 30.479 \text{ cm.}}{60 \text{ sec.}} = 30.479$ cm. per sec.

(*b*) $\frac{1 \text{ km.}}{1 \text{ hr.}} = \frac{100000 \text{ cm.}}{3600 \text{ sec.}} = 27.7$ cm. per sec.

(*c*) $\frac{36 \text{ ft.}}{1 \text{ sec.}} = \frac{12 \text{ yds.}}{\frac{1}{60} \text{ min.}} = 720$ yds. per min.

(*d*) $\frac{10 \text{ yds.}}{1 \text{ sec.}} = \frac{10 \times 0.0091438 \text{ km.}}{\frac{1}{3600} \text{ hr.}} = 32.9177$ km. per hour.

(12) *Compare the magnitudes* (*a*) *of the foot per second and the mile per hour;* (*b*) *of the mile per hour and the yard per minute.*

Ans. (*a*) $\frac{\frac{1 \text{ mile}}{1 \text{ hr.}}}{\frac{1 \text{ ft.}}{1 \text{ sec.}}} = \frac{5280 \text{ ft.}}{3600 \text{ sec.}} \times \frac{1 \text{ sec.}}{1 \text{ ft.}} = \frac{5280}{3600} = \frac{22}{15}$. Hence 1 mile per hour is to 1 ft. per sec. as $\frac{22}{15}$ to 1, or as 1.466 to 1.

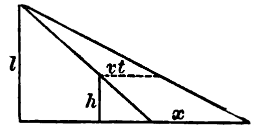
(*b*) $\frac{\frac{1 \text{ mile}}{1 \text{ hr.}}}{\frac{1 \text{ yd.}}{1 \text{ min.}}} = \frac{1760 \text{ yds.}}{60 \text{ min.}} \times \frac{1 \text{ min.}}{1 \text{ yd.}} = \frac{1760}{60} = 29\frac{1}{3}$. Hence 1 mile per hour is to 1 yd. per min. as $29\frac{1}{3}$ to 1.

(13) *A point describes* 50 *feet in* 6 *minutes and another point describes* 50 *centimeters in* 6 *seconds. Compare their mean speeds.*

Ans. $\frac{\frac{50 \text{ ft.}}{6 \text{ min.}}}{\frac{50 \text{ cm.}}{6 \text{ sec.}}} = \frac{50 \times 30.47945 \text{ cm.}}{360 \text{ sec.}} \times \frac{6 \text{ sec.}}{50 \text{ cm.}} = \frac{30.47945}{60} = 0.508$. Hence the mean speed in the first case is to that in the second as 0.508 to 1.

(14) *A man h feet in height walks along a level street at a uniform*

speed of v miles per hour, in a straight line from an electric light l ft. in height. Find the mean speed of the end of his shadow.



Ans. $l - h : vt :: l : x$.

Hence $\frac{x}{t} = \frac{l}{l-h}v =$ speed required.

(15) *A passenger in a railroad car moving with uniform speed counts 50 telegraph poles at equal intervals of 100 ft. passed in one minute. What is the mean speed of the train?*

Ans. 56.8 miles per hour.

(16) *Three planets describe paths which are to each other as 15, 19, and 12, in times which are as 7, 3, and 5. Find their comparative speeds.*

Ans. 225, 665, and 252.

(17) *Two bodies A and B describe the same path in the same direction, with uniform speeds v and v′, and at the start the distance between them is a. Find the time t when they will be at the distance b in the path, and the distance of each from the initial position of A at the end of that time.*

Ans. Take, as the example requires, the position of A, when $t = 0$, as the origin, and let distances in front of this origin be (+) and behind it be (−). Then for the distance of A from the origin at the end of any time t we have $s = vt$. For the distance of B from the origin at the end of the same time, if B is initially in front of the origin A, we have $s' = a + v't$; if B is initially behind the origin A, $s' = -a + v't$. In general, then, $s' = v't \pm a$, where the (+) sign is taken for a, when B is initially in front, and the (−) sign when B is initially behind the origin A. We have, then, $s - s' = vt - v't \mp a$.

But by the conditions of the problem $s - s' = b$. Hence

$$b = vt - v't \mp a, \quad \text{or} \quad t = \frac{b \pm a}{v - v'}.$$

Substituting this value of t, we have

$$s = v\frac{b \pm a}{v - v'}, \quad s' = \pm a + v'\frac{b \pm a}{v - v'} = \frac{v'b \pm av}{v - v'},$$

where the (+) sign or (−) sign for a is taken according as B starts ahead of or behind A.

(18) *In the preceding example, suppose the bodies move in the path in opposite directions.*

Ans. For the distance of A from the origin at the end of any time t, we have, as before, $s = vt$. If B is initially in front of A and moves in the opposite direction, we have $s' = a - v't$. *If B is initially behind A, we have* $s' = -a - v't$. In both cases, then, $s' = -v't \pm a$, where, as before, the (+) sign is taken for a when B is initially in front of A, and the (−) sign when B is initially behind A. We have, then, $s - s' = b = vt + v't \mp a$; hence

$$t = \frac{b \pm a}{v + v'}, \quad s = v\frac{b \pm a}{v + v'}, \quad s' = \frac{\pm av - v'b}{v + v'}.$$

(19) *Required the time when the two bodies are together.*

Ans. In this case $b = 0$, and

$$t = \frac{\pm a}{v \pm v'}, \quad s = s' = \frac{\pm av}{v \pm v'},$$

where the (+) sign or (−) sign is to be taken for a according as B starts ahead or behind A, and the (+) sign or (−) sign for v' according as the bodies move in opposite or in the same directions in the path.

(20) *Give finally the general solution.*

Ans. $t = \frac{b \pm a}{v \pm v'}$, $s = v\frac{b \pm a}{v \pm v'}$, $s' = \frac{\pm av \pm v'b}{v \pm v'}$,

where a has the (+) or (−) sign according as B is in front or behind A, and v' has the (+) or (−) sign according as the bodies move in opposite or in the same directions in the path.

(21) *Two bodies A and B move in the circumference of a circle of length c, with uniform speeds v and v', the distance apart at the beginning of the time being a. Find the time of the nth meeting, the space described by A and B, and the interval between two successive meetings.*

Ans. Placing $b = 0$ in the value for t in example 20, we have

$$t_1 = \frac{\pm a}{v \pm v'} = \text{time of the first meeting.}$$

Then $$t_2 = \frac{\pm a + c}{v \pm v'} = \text{time of the second meeting,}$$

$$t_3 = \frac{\pm a + 2c}{v \pm v'} = \text{time of the third meeting.}$$

In general, $$t_n = \frac{\pm a + (n-1)c}{v \pm v'} = \text{time of the } n\text{th meeting.}$$

Also, $$s = vt_n = v\frac{\pm a + (n-1)c}{v \pm v'} = \text{space described by } A,$$

and $$s \mp a = \frac{\pm av' + v(n-1)c}{v \pm v'} = \text{space described by } B.$$

The interval between two successive conjunctions is

$$t_2 - t_1 = t_3 - t_2 = \frac{c}{v \pm v'}.$$

We take (+) or (−) sign for a according as B is in front of or behind A at start, and (+) or (−) sign for v' according as the bodies move in opposite or same directions in the path.

(22) *When the earth is in that part of its orbit nearest to Jupiter, an eclipse of one of Jupiter's satellites is seen* 16 *min.* 30 *sec. sooner than when the earth is most remote from Jupiter. The radius of the earth's orbit being* 92390000 *miles, what is the speed of light?*

Ans. 186000 miles per sec.

(23) *A train of cars moving with a speed of* 20 *miles an hour had been gone three hours, when a locomotive was dispatched in pursuit, with a speed of twenty-five miles an hour. Find the time of meeting, the speeds being maintained uniform during the time.*

Ans. $t = 12$ hours.

(24) *Had the trains in the preceding example started together and moved in opposite directions around the earth,* 24840 *miles, in what time would they meet?*

Ans. 23 days. (See example 21.)

(25) *It is just one o'clock by a clock. Find the time elapsed when the minute and hour hands will be together.*

Ans. $5\frac{5}{11}$ minutes. (See example 21.)

(26) *The daily motion of Mercury in his orbit is* 4°.09239; *that of*

Venus 1°.60216; *that of the earth* 0°.98563. *What are the intervals between the epochs at which Mercury and Venus respectively will be in the same direction from the sun as the earth?*

Ans. 115.876 days and 583.913 days. (See example 21.)

(27) *A man caught in a shower in which the rain fell vertically, ran with a speed of* 12 *feet per sec. He found that the drops appeared to strike his face at an angle of* 10° *with the vertical. What was the speed of the drops?*

Ans. 12 tan 80° = 68 ft. per sec.

(28) *When the path of the earth in its orbit is perpendicular to a line drawn from a star to the earth, the direction of the star appears to make an angle of* 20″.445 *with the perpendicular to the path of the earth. The speed of the earth being* 68180 *miles per hour, what is the speed of light?*

Ans. 191030 miles per sec.

(29) *Compare the speeds of two locomotives, one of which travels* $397\frac{4}{5}$ *miles in* $11\frac{1}{4}$ *hours and the other* $262\frac{4}{13}$ *miles in* $8\frac{1}{3}$ *hours.*

Ans. 91 to 81.

(30) *A car makes a circuit of* 4½ *miles in one hour, stopping at two stations five minutes and two minutes respectively, and making twenty other stoppages of an average duration of* 10 *sec. each; find the average speed.*

Ans. 5.44 miles per hour.

(31) *An ordinary train takes ten hours to a certain trip, besides two hours in all of stoppages. The express goes* 50 *per cent faster, and makes the trip in* 4 *hours less. What time does it lose in stoppages?*

Ans. 1 hour 20 min.

(32) *A man rides a certain distance and walks back in six hours. He could ride both ways in* 3½ *hours. How long would it take him to walk both ways?*

Ans. 8½ hours.

(33) *The speed of the periphery of a wheel* 12 *feet in diameter is* 6 *feet per sec.; find the revolutions per minute.*

Ans. 9.5 rev. per min.

(34) *A person inquiring the time of day is told that it is between* 5 *and* 6 *o'clock, and that the hour and minute hands are together. Find the time.*

Ans. 5 hr. 27 min. $16\frac{4}{11}$ sec.

(35) *Find the number of revolutions per mile made by a wheel* 4½ *feet diameter.*

Ans. 373 rev. per mile.

(36) *How soon after* 8 *o'clock are the hour and minute hands directly opposite?*

Ans. $10\frac{10}{11}$ min.

(37) *Two men walk opposite ways round a circular course. They meet for the first time at the north point, the sixth time at the east point. Where will they meet for the sixteenth time, and what are the relative speeds?*

Ans. At the west point. Ratio of the speeds, 19 to 1, or 3 to 1.

(38) *A courier starts from a given point with a speed of b miles in a hours. After n hours a second courier, travelling at the rate of d miles in c hours, sets out from a point q miles ahead or behind the first point, and travels over the same route. In what time will the second courier overtake the first?* (See example 19.)

Ans. $\frac{nb \pm aq}{ad - cb}c$ hours, where the (−) sign is taken if q is ahead and the (+) sign if q is behind.

(39) *A hollow ball A floating upon the surface of a stream is observed to have a speed of* v_0. *If A is united by a thin wire to another ball B which sinks in water, the speed of A is now observed to be* v_1. *Find the speed* v_1 *of the water at the depth of B.*

Ans. We have the combined speed $v = \frac{v_0 + v_1}{2}$, hence $v_1 = 2v - v_0$.

CHAPTER II.

RATE OF CHANGE OF SPEED.

EQUATIONS OF MOTION UNDER DIFFERENT RATES OF CHANGE OF SPEED. GRAPHIC REPRESENTATION OF RATE OF CHANGE OF SPEED.

Change of Speed.—When the speed of a point is variable, the difference between the final and initial instantaneous speeds for any interval of time is the total or *integral* change of speed for that time.

Mean Rate of Change of Speed.—The integral change of speed *per unit of time* is the **mean rate of change of speed.** Therefore the number of units in the integral change of speed for any interval of time, divided by the number of units in that time, gives the number of units in the mean rate of change of speed.

When the mean rate of change of speed varies with the interval of time, it is **variable.** When it has the same magnitude, no matter what the interval of time, it is **uniform.**

Instantaneous Rate of Change of Speed.—The limiting value of the mean rate of change of speed, when the interval of time is indefinitely small, is the **instantaneous rate of change of speed.**

Rate of change of speed should always be understood as meaning the instantaneous rate of change unless otherwise specified.

Rate of change of speed may be zero, uniform or variable. When it is zero, the speed is uniform and is the same as the mean speed for any interval of time.

When it is uniform, the rate of change of speed is the same as the mean rate of change for any interval of time.

When it is variable, the mean rate of change has different values for equal intervals of time.

Dimensions of Unit of Rate of Change of Speed.—Let us denote the rate of change of speed by A and its unit by $[A]$ and the number of units by a, so that $A = a[A]$. Then if $v_1[V]$ and $v[V]$ are the initial and final instantaneous speeds and $t[T]$ the corresponding interval of time, we have the integral change of speed $(v - v_1)[V]$, and by definition the mean rate of change of speed is

$$a[A] = \frac{(v - v_1)[V]}{t[T]}.$$

We shall always have the numeric equation

$$a = \frac{v - v_1}{t}$$

if we take

$$[A] = \frac{[V]}{[T]} = \frac{[L]}{[T]^2} \text{ (page 16).}$$

The unit of rate of change of speed is then one unit of speed per unit of time, as for instance one foot-per-sec. per sec.

Numeric Equations of Rate of Change of Speed.—We have then the numeric equation for the mean rate of change of speed

$$a = \frac{v - v_1}{t}, \qquad (1)$$

where v and v_1 are the number of units in the final and initial instantaneous speeds during the interval of time t units.

For the instantaneous rate of change of speed we have the limiting value in the notation of the Calculus

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \qquad (2)$$

Sign of Rate of Change of Speed.—From (1) we see that if v is numerically greater than v_1 the value of a will be positive (+), and if v_1 is numerically greater than v the motion is retarded and the value of a is negative (—). The value of a then is positive (+) when the speed increases, and negative (—) when the speed decreases during the interval of time t.

Rate of Change of Speed a Scalar Quantity.—It is evident that the sign of a has no reference to any special direction, but simply shows whether the speed is increasing or decreasing numerically. Rate of change of speed is therefore, like speed (page 16), a *scalar* quantity.

The student is cautioned here not to confound rate of change of speed with "acceleration," which, as we shall see hereafter (page 49), has direction as well as magnitude and sign, and is therefore a vector quantity.

EXAMPLES.

(1) *A point moves at a given instant with a speed of* 12 *ft. per sec., and at the end of* 3 *sec. after, with a speed of* 16 *ft. per sec. What is the change of speed, and the mean rate of change of speed?*

Ans. + 4 ft. per sec.; + $\frac{4}{3}$ ft.-per-sec. per sec. The (+) sign indicates that the speed increases.

(2) *A train starts from rest and in* 8 *min. attains a speed of* 30 *miles an hour. What is the uniform rate of change of speed?*

Ans. + 225 miles-per-hour per hour.

(3) *Compare the foot-per-sec. per sec. and the yard-per-min. per min.*

Ans. 1 ft.-per-sec. per sec. is equivalent to 1200 yards-per-min. per min.

(4) *Reduce* 400 *ft.-per-min. per min. to kilometers-per-hour per hour, to feet-per-sec. per sec., and to centimeters-per-sec. per sec.*

Ans. 438.9 kil.-per-hour per hour; $\frac{1}{9}$ ft.-per-sec. per sec.; 3.38 cen.-per-sec. per sec.

(5) *Reduce* 1 *mile-per-min. per min. to centimeters-per-sec per sec.*

Ans. 44.7.

(6) *If we take* 32.2 *ft.-per-sec. per sec. as the unit of rate of change of speed, and* 5 *yards as the unit of length, what should be the unit of time?*

Ans. $[T] = \sqrt{\frac{[L]}{[A]}} = \sqrt{\frac{15 \text{ ft.}}{32.2 \text{ ft.}} 1 \text{ sec.}^2} = 0.68$ sec.

(7) *In 15 sec. the speed of a point changes from 400 to 100 ft. per sec. Find the uniform mean rate of change of speed.*

Ans. − 20 ft.-per-sec. per sec. The minus sign shows that the speed decreases.

(8) *The distance measured in the path of a moving point from a fixed point is given by $s = at + bt^2$, where s is the number of feet passed over in the number of seconds t. What is the mean speed, instantaneous speed, mean rate of change of speed, instantaneous rate of change of speed?*

Ans. Mean speed, $\frac{s - s_1}{t - t_1} = a + b(t + t_1)$. Instantaneous speed, $\frac{ds}{dt} = a + 2bt$. Mean rate of change of speed, $\frac{v - v_1}{t - t_1} = 2b$. Instantaneous, the same.

(9) *If the distance is given by $s = 2t + 3t^2 + 4t^3$, what is the mean speed between the origin and final position at the end of 5 sec.? What is the instantaneous speed at the origin and at the end of 5 sec.? What is the mean rate of change of speed between the origin and at the end of 5 sec.? What is the instantaneous rate of change of speed at the origin and at the end of 5 sec.?*

Ans. Mean speed, $\frac{s - s_1}{t - t_1} = 2 + 3(t + t_1) + 4(t^2 + tt_1 + t_1^2)$. If we let $s_1 = 0$ when $t_1 = 0$, the time t counts from the origin, and we have for $t = 5$ mean speed = 117 ft. per sec.

Instantaneous speed, $v = \frac{ds}{dt} = 2 + 6t + 12t^2$. For $t = 0$ this gives 2 ft. per sec. For $t = 5$, 332 ft. per sec.

Mean rate of change of speed, $\frac{v - v_1}{t - t_1} = 6 + 12(t + t_1)$. For $t_1 = 0$ and $t = 5$, this gives 66 ft.-per-sec per sec.

Instantaneous rate of change of speed, $\frac{dv}{dt} = 6 + 24t$. For $t = 0$ this gives 6 ft.-per-sec. per sec. For $t = 5$, 126 ft.-per-sec. per sec.

(10) *If a point traverses in t units of time a distance measured in the path of s units of length, given by $s = \frac{a}{t} + bt^2$, where a and b are constants, what is the mean speed, the instantaneous speed, the mean rate of change of speed, the instantaneous rate of change of speed?*

Ans. Mean speed, $\frac{s - s_1}{t - t_1} = -\frac{a}{tt_1} + b(t + t_1)$.

Instantaneous speed, $v = \frac{ds}{dt} = -\frac{a}{t^2} + 2bt$.

Mean rate of change, $\frac{v - v_1}{t - t_1} = \frac{a(t + t_1)}{t_1^2 t^2} + 2b$.

Instantaneous rate of change, $\frac{dv}{dt} = \frac{2a}{t^3} + 2b$.

(11) *A steamer leaves Liverpool for New York, and a vessel leaves New York for Liverpool at the same time; they meet, and when the steamer reaches New York, the vessel has as far to go as the steamer had when they met. If the distance is 3000 miles, how far out from Liverpool did they meet?*

Ans. 1854 miles.

(12) *A walks at the speed of 3¾ miles per hour and starts 18 minutes before B. At what speed must B walk to overtake A at the ninth milestone ?*

Ans. 4.29 miles per hour.

(13) *A tourist left behind by his companions, wishes to rejoin them on the following day. He knows they are 5 miles ahead, will start in the morning at 8 o'clock, and will walk at the rate of 3¼ miles an hour. When must he start in order to overtake them at 1 o'clock p.m., walking at the rate of 4 miles an hour, and resting once for half an hour on the road ?*

Ans. 7 h. 12 m. A.M.

(14) *A man walking 4 miles an hour meets 20 street cars in an hour and is overtaken by 4. What is the average speed of the cars, and what is the average distance between successive cars ?*

Ans. 6 miles per hour ; ½ mile.

(15) *A starts from a railway station, walking 5 miles an hour ; at the end of an hour B starts, walking 4 miles an hour. At the end of another hour a train starts and passes A 25 minutes after it passes B. Find the speed of the train.*

Ans. 20 miles an hour.

(16) *A passenger-train going 41 miles an hour and 431 feet long overtakes a freight on a parallel line. The freight-train is 713 feet long and is going 28 miles an hour. How long does it take the passenger-train to pass ?*

Ans. 1 minute.

(17) *In a mile race A gives B 50 yards. B passes the line 5 minutes after the start. A passes it 5 seconds later. Which would win in an even race, and by what distance ?*

Ans. *A* by 21¼ yards.

Equations of Motion of a Point under Different Rates of Change of Speed.—The rate of change of speed may be zero, uniform or variable. When variable, it may vary according to any law.

(*a*) **Rate of Change of Speed Zero.**—When the rate of change of speed is zero, the speed is uniform, and the instantaneous speed at any instant is equal to the mean speed for any interval of time.

In this case, if s_1 is the number of units in the initial distance, measured along the path from the origin, and s the number of units in the final distance, we have

$$v = \frac{s - s_1}{t}, \quad \text{or} \quad vt = s - s_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation will be general if we take distances from the origin in one direction as (+) and in the other direction as (—). In such case, if the value of v comes out (+) it denotes motion in the assumed (+) direction; if (—), it denotes motion in the other direction. If t comes out (+) it denotes time after, if (—) time before the start, or beginning of motion.

(*b*) **Rate of Change of Speed Uniform.**—When the rate of change of speed is uniform, the instantaneous rate of change of speed at any instant is equal to the mean rate of change of speed for any interval of time.

In this case, if v and v_1 are the number of units in the initial and final instantaneous speeds for any interval of time t units, we have for the uniform rate of change of speed

$$a = \frac{v - v_1}{t}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

The value of a is (+) when the speed increases and (—) when it decreases during the interval of time t.

From equation (2) we have

$$v = v_1 + at. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

The average speed during the interval t is

$$\textit{mean speed} = \frac{v + v_1}{2} = v_1 + \frac{1}{2}at. \quad . \ . \ . \ . \ . \quad (4)$$

The distance $(s - s_1)$ *between the initial and final positions*, described in the time t, is the mean speed multiplied by the time, or

$$s - s_1 = \frac{v + v_1}{2}t = v_1 t + \frac{1}{2}at^2. \quad . \ . \ . \ . \quad (5)$$

Inserting the value of t from (2) we have

$$s - s_1 = \frac{v^2 - v_1^2}{2a}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (6)$$

Hence $$v^2 = v_1^2 + 2a(s - s_1). \quad . \ . \ . \ . \ . \ . \ . \quad (7)$$

In applying these formulas, a is positive (+), when the speed increases, and negative (—), when the speed decreases, without regard to direction of motion.

If distances s, s_1, from the origin, in one direction are taken as positive (+), distances in the opposite direction are negative (—).

Speeds v, v_1 are positive (+) when motion is in the assumed positive direction, and negative (—) when in the other direction. If t is minus (—), it denotes time before the beginning of motion; if plus (+), time after.

By means of these equations, if we have given the initial position of a point moving in any path, its initial speed and uniform change of speed, we can determine its final position and speed and the distance described in any given interval of time.

[(*c*) **Rate of Change of Speed Variable.**]—If the rate of change of speed is variable, we have from (1), in Calculus notation, for the instantaneous speed,

$$v = \frac{ds}{dt}; \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

and from (2),

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}; \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

and from (8),

$$s - s_1 = \int_{t=0}^{t=t} v\,dt. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (10)$$

The preceding equations for constant rate of change of speed can be directly deduced from these three general equations.

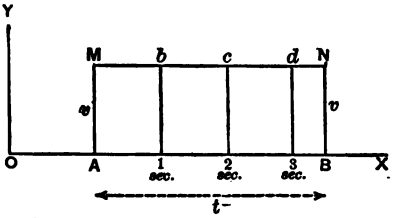
Thus if we suppose a constant, we have, by integrating (9), $v = at + C$, where C is the constant of integration. When $t = 0$, we have $v = v_1$, and hence $C =$

v_1. Hence $v = v_1 + at$, which is equation (3). Inserting this value of v in (8) and integrating, we have $s = v_1t + \frac{1}{2}at^2 + C$. But when $t = 0$ we have $s = s_1$, hence $C = s_1$, and, therefore, $s - s_1 = v_1t + \frac{1}{2}at^2$. This is equation (5).

In any case, if the law of variation of a is given, we can find the relation between v, s and t.

Graphic Representation of Rate of Change of Speed.—If we represent intervals of time by distances laid off horizontally along the axis of x, and the corresponding speeds by ordinates parallel to the axis of y, we shall have in general a curve for which the change of y with x will show the law of change of speed with the time.

(*a*) **Rate of Change of Speed Zero.**—Lay off from A along AB equal distances, so that the distances from A to 1, 1 to 2, 2 to 3, etc., are all equal and represent each one second of time, and let AB represent the entire time t.

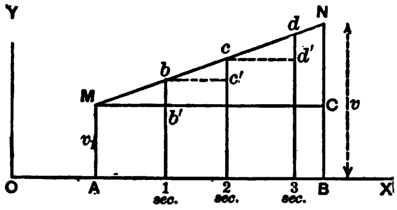


Then at A, 1, 2, 3, and B erect the perpendiculars AM, $1b$, $2c$, $3d$, BN, and let the length of each represent the speed at the corresponding instant.

Since there is no change of speed, these perpendiculars will all be of equal length, we shall have $AM = 1b = 2c = 3d = BN = v$, and the speed at any interval of time will be given by the ordinate at that instant to the line MN parallel to AB.

The space described in any time is given by $s - s_1 = vt$. This is evidently given by the area $AMNB$ in the diagram. Therefore, *the area corresponding to any time gives the space described in that time.*

(*b*) **Rate of Change of Speed Constant.**—Lay off as before the time along AB, and at A, 1, 2, 3, B, the corresponding speeds, so that AM is the initial speed v_1 and BN the final speed v. Draw MC, bc', cd', parallel to AB.



Then bb' will be the change of speed in the first sec., cc' the change of speed in the next sec., and so on. Since these are to be constant, NM is a straight line, the ordinate to which at any instant will give the speed at that instant.

The rate of change of speed is then $\frac{bb'}{1 \text{ sec.}} = \frac{cc'}{1 \text{ sec.}} = \frac{dd'}{1 \text{ sec.}} = a$.

But $\frac{bb'}{1 \text{ sec.}} = \frac{v - v_1}{t} = a$. *Hence the rate of change of speed is the tangent of the angle NMC which the line MN makes with the horizontal.* Hence $a = \frac{NC}{t}$ or $NC = at$.

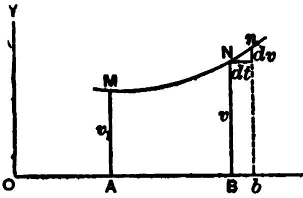
The distance described in the time t is from equation (5) given by $\frac{v + v_1}{2}t$. But this is the area of $AMNB$. Therefore, *the area corresponding to any time gives the space described in that time.*

We have then directly from the figure, since $NC = at$,

$$s - s_1 = \frac{v + v_1}{2}t = v_1t + \frac{1}{2}NC \times t = v_1t + \frac{1}{2}at^2.$$

If v_1 is greater than v, a will be negative, and the line *MN* is inclined below the horizontal *MC*.

[(*c*) **Rate of Change of Speed Variable.**]—If the rate of change of speed is not constant, we shall have in general a curve *MNn*. The tangent to this curve at any point *N* makes an angle with the axis of *X*, whose tangent is $\frac{dv}{dt} = a$, equation (9), or the rate of change of speed. The elementary area $BNnb = vdt = ds$, equation (8), and the total area $AMNB = \int_{t=0}^{t} vdt = s - s_1$, equation (10).



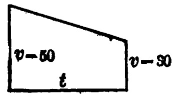
When $\frac{dv}{dt} = 0$, or $\frac{d^2s}{dt^2} = 0$, or $a = 0$, the tangent to the curve is horizontal at the corresponding point, and we have the speed at that point a maximum or minimum, according as the curve is concave or convex to the axis of *X*.

EXAMPLES.

(1) *The speed of a point changes from 50 to 30 ft. per sec. in passing over 80 ft. Find the constant rate of change of speed and the time of motion.*

Ans. $a = \frac{v^2 - v_1^2}{2(s - s_1)} = \frac{900 - 2500}{2 \times 80} = -10$ ft.-per-sec. per sec. The minus sign indicates decreasing speed. $t = \frac{v - v_1}{a} = \frac{30 - 50}{-10} = 2$ sec.

(2) *Draw a figure representing the motion in the preceding example, and deduce the results directly from it.*



Ans. Average speed $= \frac{50 + 30}{2} = 40$ ft. per sec. Hence $40t = 80$, or $t = 2$ sec. Also $a = \frac{30 - 50}{2} = -10$ ft.-per-sec. per sec.

(3) *A point starts from rest and moves with a constant rate of change of speed. Show that this rate is numerically equal to twice the number of units of distance described in the first second.*

Ans. We have $t = 1$ and $v_1 = 0$; hence from eq. (5) $\frac{s - s_1}{1 \text{ sec.}^2} = \frac{1}{2}a$, or $a = \frac{2(s - s_1)}{1 \text{ sec.}^2}$, which is *numerically* equal to $2(s - s_1)$.

(4) *In an air-brake trial, a train running at* 40 *miles an hour was stopped in* 625.6 *ft. If the rate of change of speed was constant during stoppage, what was it?*

Ans. From eq. (6), we have for $v = 0$, $s - s_1 = 625.6$, and $v_1 = \frac{40 \times 5280}{60 \times 60}$, $a = -\frac{v_1^2}{2 \times 625.6} = -\frac{(40 \times 5280)^2}{(60 \times 60)^2 \times 2 \times 625.6} = -2.75$ ft.-per-sec. per sec. The (—) sign shows retardation.

(5) *A point starts with a speed v_1 and has a constant rate of change of speed $-a$. When will it come to rest, and what distance does it describe?*

Ans. From eq. (3), when $v = 0$, we have $v_1 - at = 0$, or $t = \frac{v_1}{a}$. From eq. (5) $s - s_1 = \frac{v_1}{2}t = \frac{v_1^2}{2a}$.

(6) *A point describes* 150 *ft. in the first three seconds of its motion and* 50 *ft. in the next two seconds. If the rate of change of speed is constant, when will it come to rest? When will it have a speed of* 30 *ft. per sec.?*

Ans. From eq. (5) we have for $s - s_1 = 150$ and $t = 3$, $150 = 3v_1 + \frac{9}{2}a$; and for $s - s_1 = 200$ and $t = 5$, $200 = 5v_1 + \frac{25}{2}a$. Combine these two equations and we have $a = -10$ ft.-per-sec. per sec., and $v_1 = 65$ ft. per sec. From eq. (3), if $v = 0$, we have $65 - 10t = 0$, or $t = 6.5$ sec. From eq. (3) we also have, if $v = 30$, $30 = 65 - 10t$, or $t = 3.5$ sec.

(7) *A point whose speed is initially* 30 *meters per sec. and is decreasing at the rate of* 40 *centimeters-per-sec. per sec., moves in its path until its speed is* 240 *meters per minute. Find the distance traversed and the time.*

Ans. We have $v_1 = 30$ and $v = 4$ meters per sec., and $a = -0.4$ meters-per-sec. per sec. From eq. (6) $s - s_1 = \frac{16 - 900}{-0.8} = 1105$ meters. From (3) we have $4 = 30 - 0.4t$, or $t = 65$ sec.

(8) *A point has an initial speed of v_1 and a variable rate of change of speed given by $+ kt$, where k is a constant. What is the speed and distance described at the end of a time t?*

Ans. From eq. (9) we have $a = \frac{dv}{dt} = kt$, and integrating, $v = \frac{kt^2}{2} + C$. If, when $t = 0$, we have $v = v_1$, we obtain $C = v_1$, and hence $v = v_1 + \frac{kt^2}{2}$.

From eq. (8) $ds = vdt = v_1dt + \frac{kt^2dt}{2}$. Integrating, $s = v_1t + \frac{kt^3}{6} + C$. If, when $t = 0$, we have $s = 0$, we obtain $C = 0$, and hence $s = v_1t + \frac{kt^3}{6}$.

(9) *A point has an initial speed of* 60 *ft. per sec. and a rate of change of speed of* + 40 *ft.-per-sec. per sec. Find the speed after* 8 *sec.; the time required to traverse* 300 *ft.; the change of speed in traversing that distance; the final speed.*

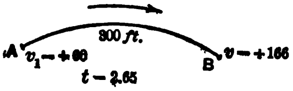
Ans. From eq. (3) we have $v = 60 + 40 \times 8 = 380$ ft. per sec. From eq. (5) we have $300 = 60t + 20t^2$, or $t = \pm\sqrt{\frac{69}{4}} - \frac{3}{2} = +2.65$ sec. or -5.65 sec. The first value only applies.

From eq. (3) we have $v - v_1 = 40t = 20(\pm\sqrt{69} - 3) = +106$ ft. per sec. or -226 ft. per sec. The first value only applies.

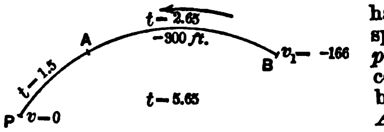
We have for the final speed $v = +166$ ft. per sec. or -166 ft. per sec. The first value only applies.

That is, the point starts from A with the speed $v_1 = 60$ ft. per sec. and describes the path $AB = 300$ ft. in $t = 2.65$ sec., the speed at B being $v = 166$ ft. per sec.

In order to interpret the negative values obtained, we observe that $v = -166$ ft. per sec. means that the point moves in the opposite direction.



Let the point then start from B in the *opposite direction* with the speed $v_1 = -166$ ft. per sec. Then from eq. (3) we have $v = -166 + 40t$. We see that when $t = 4.15$ sec., $v = 0$, and the point has passed to some point P, where the speed is zero. This point is the *turning-point*. For t greater than 4.15 sec. v becomes positive; that is, the point moves back towards B, and arrives at a point A where the speed is $v = +60$ in the time given by $60 = 40t$, or $t = 1.5$ sec. The entire time from B to P and back to A is $t = 5.65$ sec. This is the time given by the negative value of t in the example; that is, it is the *time before* the start, during which the point moves from B to P and back to A. The change of speed $v - v_1$ is $+60 + 166 = +226$, which is the negative value in the example.

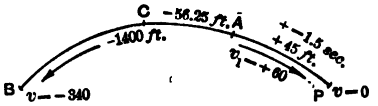


For the space BA described *between the initial and final positions*, we have $\frac{v + v_1}{2}t$ or $\frac{+60 - 166}{2} \times 5.65 = -300$ ft., the (–) sign showing that the distance is on the other side of the origin, from the case of the example.

We see then that our equations are general if we have regard to the signs of v, v_1, s, s_1, and a.

(10) *If the motion in example* 9 *is retarded, find* (*a*) *the distance described from the starting to the turning point;* (*b*) *the distance described from the starting-point after* 10 *sec., the speed acquired and the distance between the final and initial positions;* (*c*) *the distance described during the time in which the speed changes to* – 90 *ft. per sec., and this time;* (*d*) *the time required by the moving point to return to the starting-point.*

Ans. The initial speed is $v_1 = +60$ ft. per sec., the rate of change of speed is $a = -40$ ft.-per-sec. per sec. Let the time count from the start at A, so that $s_1 = 0$, when $t = 0$, and let distances and motion from A towards P be positive.



(*a*) We have from eq. (6) for the distance from A to the turning-point P, where $v = 0$,

$$s = \frac{-v_1^2}{2a} = \frac{-3600}{-80} = +45 \text{ ft.}$$

(*b*) From eq. (5) we have for the distance between the initial and final positions after 10 sec. $AB = s = v_1t + \frac{1}{2}at^2 = 60 \times 10 - 20 \times 100 = -1400$ ft. The minus sign shows distance on left of A. The total distance described is then 1490 ft. The speed acquired is given by eq. (5) $-1400 = \frac{v + 60}{2} \times 10$ or $v = -340$ ft.-per-sec. The minus sign shows motion from A towards B.

(*c*) From eq. (6) the distance between the initial and final positions, when the speed is – 90 ft. per sec., is $AC = s = \frac{v^2 - v_1^2}{2a} = \frac{8100 - 3600}{-80} = -56.25$ ft. The minus sign shows that C is on left of A. The total distance described from the start is then $56.25 + 90 = 146.25$ ft. The time, from eq. (5), is $-56.25 = \frac{-90 + 60}{2}t$, or $t = 3.75$ sec.

(*d*) The time to reach the turning-point, as we have seen, is 1.5 sec. The time to return is, from eq. (5), $45 = \frac{60}{2}t = 1.5$ sec. The entire time to go and return is then 3 sec.

(11) *A railway train runs at a speed of* 20 *miles an hour, and its speed is increasing uniformly at the rate of* 14 *feet-per-min. per min. Find its speed after* 1½ *hours, and the distance traversed in that time.*

Ans. 14 feet-per-min. per min. = 9.54 miles-per-hour per hour. From eq. (3) $v = 20 + 9.54 \times 1.5 = 34.3$ miles per hour. From eq. (6) the distance described is $\dfrac{34.3^2 - 20^2}{2 \times 9.54} = 40.7$ miles.

(12) *A railway train moving with a speed of* 50 *miles an hour has the brakes put on, and the speed diminishes uniformly for* 1 *minute, when it is found to be* 20 *miles per hour. Find the rate of change of speed, and the distance traversed. Also the time in which it will come to rest and the distance traversed.*

Ans. From eq. (3), $a = \dfrac{v - v_1}{t} = \dfrac{20 - 50}{\frac{1}{60}} = -1800$ miles-per-hour per hour.

From eq. (6) the distance traversed is $\dfrac{400 - 2500}{-3600} = \dfrac{7}{12}$ mile. In order to come to rest, $t = -\dfrac{v_1}{a} = \dfrac{-50}{-1800} = \dfrac{1}{36}$ hour = 100 sec. The distance in coming to rest $= \dfrac{-v_1^2}{2a} = \dfrac{-2500}{-3600} = \dfrac{25}{36}$ mile.

(13) *Express a rate of change of speed of* 500 *centimeters-per-second per second, in terms of the kilometer and minute.*

Ans. 18 km.-per-min. per min.

(14) *The speed and rate of change of speed of a moving point at a certain moment are both measured by* 10, *the foot and second being the units. Find the number measuring them when the yard and minute are the units.*

Ans. 200 yards per min.; 12000 yards-per-min. per min.

(15) *What is meant when it is said that the rate of change of speed of a point is* + 10, *the units being foot and second? If the point were moving at any instant at the rate of* 7½ *feet per second, after what time would its speed be quadrupled? and what distance would it describe in that time?*

Ans. 2¼ sec.; 42.1875 ft.

(16) *A body describes distances of* 120 *yards,* 228 *yards,* 336 *yards, in successive tenths of a second. Show that this is consistent with constant rate of change of speed, and find the numerical value if the units are a minute and a yard.*

Ans. 38880000 yards-per-min. per min.

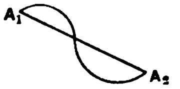
(17) *A point is moving at the rate of* 5 *feet per sec., a quarter of a minute after at the rate of* 50 *feet per sec., half a minute after at* 95 *feet per sec. Show that this is consistent with a constant rate of change of speed, and find its value.*

Ans. 3 ft.-per-sec. per sec.

CHAPTER III.

DISPLACEMENT. RESOLUTION AND COMPOSITION OF DISPLACEMENTS.

Displacement.—The total change of position, measured in units of length, of a point, between its initial and final positions, without reference to the path described, is the **linear displacement** of the point.



Thus if a point moves from the position A_1 to the position A_2, the distance A_1A_2 is the linear displacement, no matter what the path may have been from A_1 to A_2.

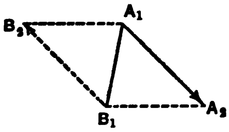
The term displacement always means linear displacement unless otherwise specified.

Line Representative of Displacement.—The displacement of a point is therefore completely represented by a straight line. The length of the line gives the magnitude of the displacement, and the direction as denoted by an arrow gives the direction of the displacement.

Thus the straight line A_1A_2 represents by its length the magnitude of a displacement, and the arrow shows that the displacement is in the direction from A_1 to A_2.

A displacement then has both magnitude and direction, and such a quantity is called a **vector quantity**. All vector quantities can be represented thus by a straight line.*

Relative Displacement.—Since the displacement of a point is change of position, it can only be determined by reference to some chosen point of reference.



Thus if A_1 and A_2 are the initial and final positions of a moving point A, and B_1 is some chosen point of reference, we call A_1A_2 the displacement of A with reference to B_1, because to an observer at B_1 the point A would move from A_1 to A_2 in the direction A_1A_2.

But to an observer on the moving point A_1 the point B_1 would appear to move from B_1 to B_2, and this is really the displacement of B relative to A.

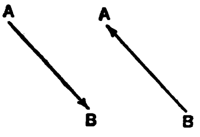
That is, *any change in the relative position of two points A and*

---

* As we shall see hereafter, linear velocity and acceleration, angular velocity and acceleration, moment of linear velocity and acceleration, moment of angular velocity and acceleration, are all vector quantities, and the same principles apply to all of them as to displacements.

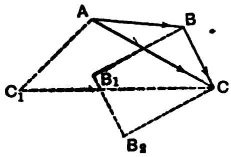
B may be regarded either as a displacement of A relative to B or as an equal and opposite displacement of B relative to A.

Relative Displacement—Two Points.—If then the straight line AB represents the displacement of a point A with reference to a point B, the equal and opposite line BA represents the displacement of B relative to A.



We shall always denote therefore relative displacement by a line, the length of which gives the magnitude of the displacement, the arrow its direction, and the letters at the end the two points. Thus in the figure one line gives the displacement of A relative to B, the other gives the displacement of B relative to A.

Relative Displacement—Three Points.—Triangle of Displacements.*—Let a moving point A have the displacement AB with reference to a point B, and at the same time let B have the displacement BC with reference to a point C_1, and let B_1, B_2 be the initial and final positions of the point B.



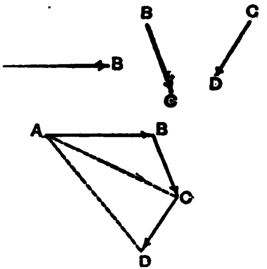
Then it is evident that since the point has the displacement AB with reference to B, and at the same time B moves from B_1 to B_2, the point moves from A to C, and AC is the displacement of A with reference to C.

Conversely, from the preceding article, CA is the displacement of C relatively to A.

Hence, *if two sides of a triangle ABC taken the same way round represent the displacements of A relative to B and B relative to C respectively, the third side taken the opposite way round will represent the displacement of A relative to C, and taken the same way round, the displacement of C relative to A.*

This is called the principle of the **triangle of displacements.** It evidently makes no difference whether the displacements are simultaneous or successive. The same principle holds true in both cases.

Polygon of Displacements.—Let AB, BC and CD be the given displacement of A relative to B, B relative to C, and C relative to D. Then if we lay off successively AB and BC in given direction and magnitude, the line AC gives the displacement of A relative to C. If we lay off CD, the line AD gives the displacement of A relative to D.



Hence, *if any number of displacements in the same plane are represented by the sides of a plane polygon, ABCD, etc., taken the same way round, the line AD which closes the polygon taken the opposite way round will give the magnitude and direction of the resultant displacement of the first point relative to the last, and taken the same way round, of the last point relative to the first.*

This principle is called the **polygon of displacements,** and it evidently holds good whether the *displacements are successive or simultaneous.*

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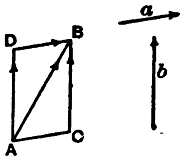
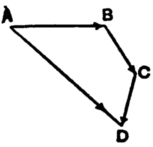
* We shall see hereafter that all the principles which follow in this chapter relating to displacements hold equally good for linear and angular velocities and accelerations, for the moments of linear and angular velocities and accelerations, and for forces and moments of forces.

Resolution of Displacements.—By the application of these principles a given displacement or any number of given displacements may be resolved into two components in any two given directions.



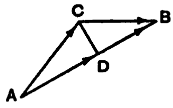
Thus suppose the displacement of A relative to B to be given by the line AB. We can resolve this displacement into components in any two directions given by the arrows a and b, by drawing lines from A and B parallel to these directions till they intersect at some point C. Then the displacements AC and CB taken the other way round are the **component** displacements of AB in the required directions.



If any number of displacements AB, BC, CD, etc., are given, we have the **resultant** displacement AD, and this displacement can be resolved as before into any two directions required.

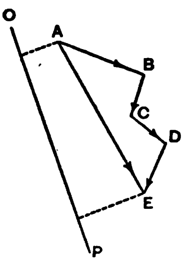
Rectangular Components.—When a displacement is thus resolved into two directions at *right angles*, the components are called **rectangular components.** Unless otherwise specified, when we speak of the components of any displacement, the rectangular components are always understood.

The Component of the Resultant is equal to the Algebraic Sum of the Components of the Displacements.—It is evident that the resultant of any two given displacements is equal to the algebraic sum of their components in the direction of the resultant.



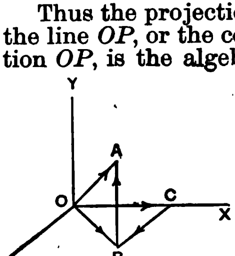
For if AC and CB are the given displacements, the resultant AB is equal to the sum of the components AD and DB.

So also for any number of displacements, the resultant AE is equal to the algebraic sum of the components of the displacements in the direction of AE.



The component in any given direction, of this resultant itself, is then equal to the algebraic sum of the components of the displacements in the same direction.

Thus the projection of the resultant AE upon the line OP, or the component of AE in the direction OP, is the algebraic sum of the components of AB, BC, CD and DE in the direction OP.

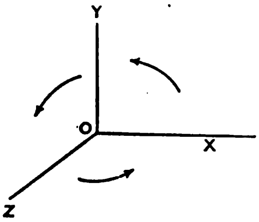


Components not in the Same Plane.—The same principles apply for components not in the same plane.

Thus if OA is a given displacement, and we draw AB perpendicular to the plane of XZ meeting it at B, we have the components OB and BA.

Again, we can resolve OB into the components OC and CB. The components then are OC, CB and BA.

Sign of Components of Displacement.—A sign of (+) or (−) prefixed to the magnitude of a component displacement indicates direction. Thus for three rectangular axes OX, OY, OZ, a component displacement in the directions OX, OY, OZ is positive, and in the opposite directions negative.

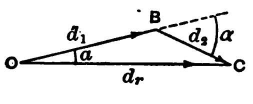


If polar co-ordinates are used, the component displacement along the radius vector is positive when away from the pole, negative when towards the pole.

Evidently, then, we measure angles in the plane XY from OX around towards OY; in the plane YZ from OY around towards OZ; in the plane ZX from OZ around towards OX, as shown by the arrows in the figure.

Analytical Determination of the Resultant for Concurring Displacements.—When the line representatives meet in a point they are called **concurring.** All displacements of a single point must be concurring. The magnitude and direction of any number of such concurring component displacements being given, to find expressions for the magnitude and direction of the resultant.

(*a*) **When there are Two Given Components.**—Let $OB = d_1$ and $BC = d_2$ be two component displacements making the angle α and in the indicated directions.



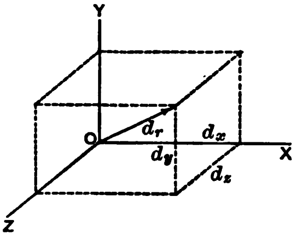
Then the resultant is $OC = d_r$ and is given at once in magnitude from the triangle OBC,

$$d_r^2 = (d_1 + d_2\cos\alpha)^2 + (d_2\sin\alpha)^2 = d_1^2 + 2d_1d_2\cos\alpha + d_2^2. \quad . \quad (1)$$

The resultant d_r makes with d_1 an angle a, given by

$$\cos a = \frac{d_1 + d_2\cos\alpha_2}{d_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(*b*) **When there are Any Number of Components in any Given Direction.**—Take three rectangular axes, OX, OY, OZ, through the point O, and let the component displacements d_1, d_2, d_3, etc., make the angles α_1, β_1, γ ; α_2, β_2, γ_2, etc., with the axes of X, Y, Z respectively.



Then we have for the sum of the components in the direction of X, Y, and Z,

$$\left.\begin{aligned} d_x &= d_1\cos\alpha_1 + d_2\cos\alpha_2 + \ldots = \Sigma d\cos\alpha; \\ d_y &= d_1\cos\beta_1 + d_2\cos\beta_2 + \ldots = \Sigma d\cos\beta; \\ d_z &= d_1\cos\gamma_1 + d_2\cos\gamma_2 + \ldots = \Sigma d\cos\gamma. \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

In these summations we must take components with their proper signs as directed in the preceding Article.

We have then for the magnitude of the resultant d_r,

$$d_r = \sqrt{d_x^2 + d_y^2 + d_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This resultant makes with the axes angles a, b, c, respectively, given by

$$\cos a = \frac{d_x}{d_r}, \quad \cos b = \frac{d_y}{d_r}, \quad \cos c = \frac{d_z}{d_r}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The projections on the planes *XY*, *YZ*, *ZX* make angles with *X*, *Y*, *Z*, respectively, whose tangents are

$$\frac{d_y}{d_x},\ \frac{d_z}{d_y},\ \frac{d_x}{d_z}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

Cor. 1. If all the displacements are in one plane, make $d_z = 0$. Then

$$d_r = \sqrt{d_x^2 + d_y^2}. \quad \ldots\ldots\ldots \quad (7)$$

$$\cos a = \frac{d_x}{d_r},\quad \cos b = \frac{d_y}{d_r}; \quad \ldots\ldots \quad (8)$$

and, since $\cos b = \sin a$,

$$\tan a = \frac{d_y}{d_x}. \quad \ldots\ldots\ldots\ldots \quad (9)$$

Cor. 2. When there are but two displacements, d_1 and d, take OX corresponding with d_1. Then $d_x = d_1 + d\cos\alpha$, $d_y = d\sin\alpha$, and we have at once equations (1) and (2).

Cor. 3. If the two displacements are in the same direction or in opposite directions, $\alpha = 0$ or 180°, and

$$d_r = d_1 \pm d.$$

That is, the resultant is the algebraic sum of the displacements.

Cor. 4. If the two displacements are at right angles, $\alpha = 90°$ and

$$d_r = \sqrt{d_1^2 + d^2},\quad \sin a = \frac{d}{d_r},\quad \sin b = \frac{d_1}{d_r} = \cos a,\quad \tan a = \frac{d}{d_1}.$$

Cor. 5. If the two displacements are equal, $d_1 = d$ and

$$d_r^2 = 2d^2(1 + \cos\alpha) = 4d^2\cos^2\frac{\alpha}{2},$$

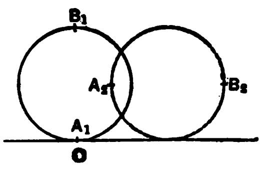
hence $$d_r = 2d\cos\frac{\alpha}{2}.$$

Also, $$\sin a = \frac{d\sin\alpha}{2d\cos\frac{\alpha}{2}} = \sin\frac{\alpha}{2},\quad \text{hence}\quad a = \frac{\alpha}{2}$$

Cor. 6. If the two displacements are equal and $\alpha = 120°$, then $d_r = d$, and $a = 60°$.

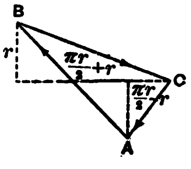
EXAMPLES.

(1) *A wheel whose radius is r rolls on a horizontal plane until it turns through a quarter revolution. Find the displacement of the point of the wheel initially in contact with the plane relative to the point diametrically opposite.*



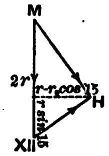
Ans. Let C represent the initial point of contact of the plane, A the point of the wheel in contact with C in the plane, B the point of the wheel diametrically opposite A.

The displacement of B with reference to C is given by B_1B_2. The displacement of A with reference to C is given by A_1A_2, and therefore the displacement of C with reference to A is given by A_2A_1. We have then, by laying off these displacements, the displacement of A relative to B equal to $2r\sqrt{2}$, making an angle of 45° with the vertical.



(2) *Find the displacement of the end of the minute-hand with reference to the end of the hour-hand of a clock, between* 3 *and* 3.30 *o'clock, the length of each hand being r.*

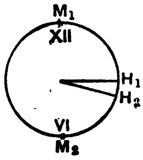
Ans. The displacement of the minute-hand with reference to XII is M_1M_2, and of the hour-hand with reference to XII, H_1H_2. Therefore the displacement of XII with reference to H is H_2H_1. Laying off these displacements, we have



$$MH = r\sqrt{6 - 2\cos 15° - 4\sin 15°},$$

and for the angle of MH with the vertical

$$\sin HM\,\text{XII} = \frac{2\sin^2 7\tfrac{1}{2}°}{\sqrt{6 - 2\cos 15° - 4\sin 15°}}.$$



(3) *A river flows in a direction N.* 33° *E., and a boat headed directly across at right angles to the current reaches a point on the other shore from which the starting-point is found to bear S.* 3° *W. If the distance from the starting-point is* 500 *ft., how far has the boat been carried by the current, and what is the distance across the river if the banks are parallel?*

Ans. The course of the boat makes an angle of 30° with the bank. The distance across is 250 ft.; the distance parallel to the bank 433 ft.

(4) *A point A moves* 30 *ft. in a given direction relative to a fixed point O. Another point B moves relative to O* 40 *ft. in a direction at right angles to A's motion. Find the displacement of A relative to B.*

Ans. 50 ft. in a direction inclined to A's motion by an angle whose sin is $\frac{4}{5}$.

(5) *Two points move in the circumference of two circles of radius r and* 2*r respectively. Both start from the point of contact. One moves through an angle of π radians, the other through an angle of* $\frac{\pi}{2}$ *radians. Find the displacement of either relative to the other.*

Ans. If one point moves through the angle π in the small circle and the other through $\frac{\pi}{2}$ in the large, the displacement is $2r\sqrt{5}$, making an angle with the vertical whose tangent is 2.

If one point moves through the angle π in the large circle and the other through $\frac{\pi}{2}$ in the small, the displacement is $r\sqrt{26}$ and it makes an angle with the vertical whose tangent is 5.

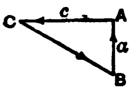
(6) *In the preceding example let the radius of the small circle be* r_1 *and of the large circle* r_2. *Find the displacement as before.*

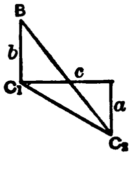
Ans. $\sqrt{4r_1^2+4r_2r_1+2r_2^2}$ and tangent $\frac{2r_2+r_1}{r_1}$,

or

$\sqrt{4r_2^2+4r_2r_1+2r_1^2}$ and tangent $\frac{2r_2+r_1}{r_1}$.

(7) *The displacement of A relative to B is a distance a towards the south, and relative to C, c towards the west. If C is initially a distance b south of B, find the final position of C relative to B.*





Ans. We have CB for the displacement of C relative to B. Therefore if C_1 is the initial position, C_2 will be the final position, C_1C_2 being equal to CB. Hence the distance of C's final position from B is $BC_2=\sqrt{(b+a)^2+c^2}$, and the direction from B to C_2 is east of south an angle whose tangent is $\frac{c}{b+a}$.

(8) *A's displacement relative to B is a to the west. C's displacement relative to B is c in a direction* 30° *west of south. Find the displacement of A relative to C.*

Ans. Displacement is $\sqrt{a^2-ac+c^2}$. Tangent of the angle with the meridian is $\frac{2a-c}{c\sqrt{3}}$. If this is positive, the angle is west of north; if negative, east of north.

(9) *Two trains A and B start from the same point and A runs* 60 *miles north and* 50 *miles northeast. Find the displacement of B relative to A.*

Ans. 43.104 miles in a direction 34° 52′ south of east.

(11) *A point undergoes two component displacements*, 100 *feet W.* 30° *S. and* 30 *ft. N. What is the resultant ?*

Ans. 88.88 ft. 13° south of west.

(12) *Three component displacements have magnitudes represented by* 1, 2 *and* 3 *and directions given by the sides of an equilateral triangle, taken the same way around. Find the magnitude of the resultant.*

Ans. $\sqrt{3}$.

(13) *Two railway trains run on parallel roads, one* 5 *miles north, the other* 6 *miles south. Find the displacement of the last relative to the first.*

Ans. 11 miles south.

(14) *Two railway trains run, the one northeast a distance d, the other southeast the same distance. Find the displacement of the first relative to the last.*

Ans. $d\sqrt{2}$, direction north.

(15) *A point undergoes three component displacements, N.* 60° *E.* 40 *ft., S.* 50 *ft., W.* 30° *N.* 60 *ft. Find the resultant.*

Ans. $10\sqrt{3}$ ft. W.

(16) *If the diagonal AC of a quadrilateral ABCD is produced to E, so that CE is equal to AC, show that AE is the resultant in the direction AC of the displacements AD, DB, BC and AC.*

(17) *ABCD is parallelogram. E is the middle point of AB. Find*

the components, in the directions of AB and AD, of a displacement which has the direction and half the magnitude of the resultant of component displacements represented by AC and AD.

Ans. *AE* and *AD*.

(18) *Prove that the resultant of two equal displacements of magnitude a, inclined* 60°, *is equal to the resultant of a and* 2*a inclined* 120°.

(19) *Prove that if two component displacements are represented by two chords of a circle drawn from a point P in the circumference perpendicular to each other, the resultant is represented by the diameter through the point.*

(20) *To an observer in a balloon the starting-point bears N.* 20° *E., and is depressed* 30° *below the horizontal. A point at the same level as the starting-point and* 10 *miles from it is vertically below him. What are the component displacements of the balloon with reference to the starting-point?*

Ans. 9.39 miles south, 3.42 miles west, and 5.77 miles up.

(21) *A point has the component displacements in the same plane,* $d_1 = 40$ *ft.,* $d_2 = 50$ *ft.,* $d_3 = 60$ *ft., making the angles with X and Y,* $\alpha_1 = 60°$, $\beta_1 = 150°$; $\alpha_2 = 120°$, $\beta_2 = 30°$; $\alpha_3 = 120°$, $\beta_3 = 150°$. *Find the resultant displacement.*

Ans. $d_x = +20 - 25 - 30 = -35$ ft.;

$d_y = -34.64 + 43.3 - 51.96 = -43.3$ ft.;

$$d_r = \sqrt{d_x^2 + d_y^2} = 55.67 \text{ ft.}$$

$$\cos a = \frac{d_x}{d_r} = \frac{-35}{55.67}, \text{ or } a = 128° \, 57' \, 17''; \quad \cos b = \frac{d_y}{d_r} = \frac{-43.3}{55.67}, \text{ or } b = 141° \, 2' \, 43''.$$

(22) *A point has the component displacements* $d_1 = 40$ *ft.,* $d_2 = 50$ *ft.,* $d_3 = 60$ *ft., making with X, Y, Z, the angles* $\alpha_1 = 60°$, $\beta_1 = 100°$, γ_1 *obtuse;* $\alpha_2 = 100°$, $\beta_2 = 60°$, γ_2 *acute;* $\alpha_3 = 120°$, $\beta_3 = 100°$, γ_3 *acute. Find the resultant displacement.*

Ans. We find the angles γ_1 (page 12) by the equation

$$\cos^2 \gamma = -\cos(\alpha + \beta)\cos(\alpha - \beta).$$

Hence $\gamma_1 = 148° \, 2' \, 31''.7$, $\gamma_2 = 31° \, 57' \, 28''.3$, $\gamma_3 = 31° \, 57' \, 28''.3$.

$d_x = +20 - 8.6824 - 30 = -18.6824$ ft.

$d_y = -6.946 + 25 - 10.419 = +7.635$ ft.

$d_z = -33.937 + 42.421 + 50.907 = +59.391$ ft.

$$d_r = \sqrt{d_x^2 + d_y^2 + d_z^2} = 62.73 \text{ ft.} \quad \cos a = \frac{-18.6824}{62.73}, \text{ or } a = 107° \, 19' \, 36'';$$

$$\cos b = \frac{+7.635}{62.73}, \text{ or } b = 83° \, 0' \, 33''; \quad \cos c = \frac{+59.391}{62.73.}, \text{ or } c = 18° \, 46' \, 42''.$$

CHAPTER IV.

VELOCITY. RESOLUTION AND COMPOSITION OF VELOCITIES.

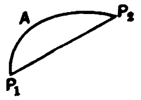
RECTANGULAR COMPONENTS OF VELOCITY. SIGN OF COMPONENTS OF VELOCITY. ANALYTIC DETERMINATION OF RESULTANT VELOCITY.

Mean Velocity.—The distance described by a moving point per unit of time we have called the mean speed of the point (page 15).

The *displacement* (page 34) of a point per unit of time we call the **mean linear velocity.** Therefore the number of units in the displacement of a point in any given time, divided by the number of units in that time, gives the number of units of mean linear velocity—or the magnitude of the mean linear velocity—while the *direction* of that velocity is the same as that of the displacement.

The term velocity always signifies linear velocity unless otherwise specified.

Mean speed, then, is mean time-rate of distance described (page 15). Mean velocity is mean time-rate of displacement.



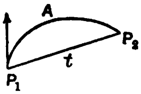
Thus if a point moves in any path P_1AP_2 from the initial position P_1 to the position P_2 in the time t units, the magnitude of the mean speed is given by $\frac{path\ P_1AP_2}{t}$, while the magnitude of the mean velocity is given by $\frac{displacement\ P_1P_2}{t}$, and its direction by the direction of P_1P_2.

If a point moves with uniform speed in a circle of radius r units and the time of revolution is t units, the mean speed is $\frac{2\pi r}{t}$ *units of speed.* But the mean velocity in the time of one revolution is zero, because the displacement in that time is zero. Again, the mean speed in one half a revolution is $\frac{\pi r}{\frac{1}{2}t}$ or $\frac{2\pi r}{t}$ *units of speed,* as before, but the mean velocity for the same time has for its magnitude $\frac{2r}{\frac{1}{2}t}$ or $\frac{4r}{t}$ *units of velocity,* and the direction is that of a diameter from the initial to the final position of the point.

Instantaneous Velocity.—The limiting magnitude of the mean speed when the interval of time is indefinitely small we have called (page 15) the **instantaneous** speed. In the same way, the limiting magnitude *and direction* of the mean velocity when the interval of time is indefinitely small is called the **instantaneous velocity**.

The terms speed and velocity should always be understood to mean instantaneous speed and instantaneous velocity unless otherwise specified.

Using the terms thus, we see that when t is indefinitely small, distance described and displacement coincide and hence the speed at P_1 is the magnitude of the velocity at P_1, while the direction of this velocity at any instant is that of the tangent to the path at that instant. *Velocity is directed speed. Speed is magnitude of velocity.*



Unit of Velocity.—Since, then, the magnitude of the velocity at any instant is the speed in the direction of the velocity at that instant, it follows that the unit of velocity is the same as the unit of speed (page 15), or one unit of length per unit of time, as, for instance, one foot per second. For this reason we have used (page 25) the letter v for the numeric for speed.

Uniform Velocity.—When the velocity has the same magnitude and direction whatever the interval of time, it is **uniform**.

Uniform velocity, then, is necessarily uniform speed in a straight line in a given direction. The velocity in such case is the same as the mean velocity for any interval of time.

Variable Velocity.—When either the magnitude or direction of the velocity changes it is **variable**.

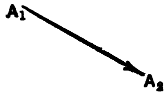
When the magnitude alone changes, we have variable speed in a straight line. When the direction only changes, we have uniform speed in a curved line. When both change, we have variable speed in a curved line. In all these cases the velocity is variable.

Thus we can speak of a point moving in a circle with uniform speed, but we cannot speak of a point moving in any curve at all with uniform velocity. If the velocity is uniform, the path must be straight, as well as speed constant.

A point can be projected with the same speed in many different directions, but we cannot speak of the same velocity in different directions A change of direction is a change of velocity whether the speed changes or not.

Velocity a Vector Quantity.—Since velocity is speed directed, or time-rate of displacement, it has not only sign and magnitude like speed (page 15), but also direction like displacement, and is therefore a **vector** quantity.

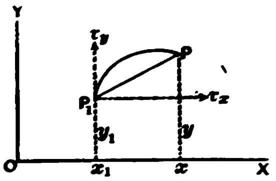
Line Representative of Velocity.—Velocity, then, can be represented like displacement (page 34) by a straight line. The length of this line represents the magnitude of the velocity, and the direction as denoted by an arrow gives the direction.



Thus the straight line A_1A_2 represents by its length the magnitude of a velocity, and the arrow shows its direction.

Triangle and Polygon of Velocities.—The principles therefore of page 35 hold good for velocities as well as displacements. We have then the **triangle and polygon of velocities.**

Resolution and Composition of Velocities.—We can also combine and resolve velocities in the same way precisely as displacements, and the principles of page 36 apply here also.



Rectangular Components of Velocity.—If a point moves in any path from P_1 to P in the time t, the displacement is the chord P_1P, and the *mean velocity* is $v = \frac{\text{chord } P_1P}{t}$.

If x_1 and y_1 are the co-ordinates of P_1, and x and y of P, then the horizontal and vertical components of the mean velocity v are

$$v_x = \frac{x - x_1}{t}, \quad v_y = \frac{y - y_1}{t}.$$

If the time is indefinitely short, we have in the notation of the Calculus, for the *instantaneous* velocities,

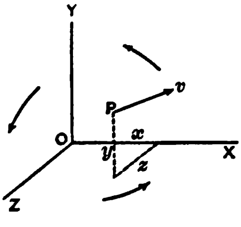
$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}.$$

Sign of Components of Velocity.—We see that if v_x is directed towards the right, we have in the 1st and 4th quadrants x numerically greater than x_1 and both are positive. In the 2d and 3d quadrants, if v_x is directed towards the right, x_1 is numerically greater than x and both are negative. In all quadrants, then, v_x will be positive when directed towards the right. In the 1st and 2d quadrants, if v_y is directed upwards, y is numerically greater than y_1 and both are positive. In the 3d and 4th quadrants, if v_y is directed upwards, y_1 is numerically greater than y and both are negative. In all quadrants, then, v_y will be positive when directed upwards.

We have then the following general rule for the signs of the components of the velocity in any quadrant:

If the direction of the line representative is towards the right, v_x is positive; if towards the left, v_x is negative. If upwards, v_y is positive; if downwards, negative.

The sign then as applied to component velocities indicates direction of motion. For rectangular co-ordinates (+) signifies towards the right or upward, (−) towards the left or downward.



For three rectangular axes OX, OY, OZ, let a point P be given by the space co-ordinates x, y, z. Let the velocity v of the point make the angles α, β and γ with the axes of X, Y and Z.

Then we have

$$v_x = v \cos \alpha, \quad v_y = v \cos \beta, \quad v_z = v \cos \gamma$$

for the components in the direction of the axes. When the angles α, β, γ are acute or less than 90°, these components are positive; when the angles are obtuse or more than 90°, these components are negative.

Therefore, as before, v_x towards the right is positive, towards the left negative, v_y upwards is positive and downwards negative, and v_z in the direction OZ is positive, in the opposite direction negative.

If polar co-ordinates are used, the component velocity along the radius vector *must be taken as positive when it acts away from the pole, and negative when it acts towards the pole.*[*]

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[*] Evidently, then, we measure angles in the plane XY from OX around towards OY; in the plane YZ from OY around towards OZ; in the plane ZX from OZ around towards OX, as shown by the arrows in the figure.

Analytical Determination of the Resultant for Concurring Velocities.—When the line representatives meet in a point they are called **concurring**. We have then the same expressions for the magnitude and direction of the resultant of any number of concurring component velocities as for displacements (pages 36 and 37). We have only to substitute v in place of d.

EXAMPLES.

(1) *To a man driving eastward with a speed of* 4 *miles an hour, the wind blows apparently from the north, but when he doubles his speed the wind appears to blow from the northeast. Find the real direction and velocity of the wind.*

Ans. The wind blows from the northeast with a velocity of $4\sqrt{2}$ miles an hour.

(2) *A point moves in a straight line from A to B*, 60 *ft. W.* 30° *S., in* 10 *sec., and thence in a straight line to C*, 30 *ft. N., in* 20 *sec. Find the mean speed and the mean velocity.*

Ans. The length of path is 90 ft. traversed in 30 sec., or mean speed is 3 ft. per sec. The displacement is $30\sqrt{3}$ ft. W., or the mean velocity is $\sqrt{3}$ ft. per sec. W.

(3) *A ship sails N.* 30° *E. at* 10 *miles an hour. Find its easterly velocity and its northerly velocity.*

Ans. 5 miles an hour; $5\sqrt{3}$ miles an hour.

(4) *A river* 1 *mile broad has a current of* 5 *miles per hour, and a boat capable of making* 10 *miles an hour in still water is to go straight across. In what direction must the boat be steered?*

Ans. Up stream in a direction inclined 60° with the bank.

(5) *Find the vertical velocity of a train moving up a* 1-*per-cent gradient at a speed of* 30 *miles per hour.*

Ans. 0.3 mile per hour.

(6) *A man travelling* 4 *miles per hour east finds the wind to come from the southeast. When he stands still it shifts* 5° *to the south. Find its velocity.*

Ans. 32.52 miles per hour N. 40° W.

(7) *A point moving with uniform speed in a circle of radius* 30 *ft. describes a quadrant in* 10 *sec. Find the mean speed, the mean velocity, the instantaneous speed and the instantaneous velocity.*

Ans. The length of path described in 10 sec. is $\frac{\pi r}{2} = 47.12$ ft. The mean speed is then 4.712 ft. per sec., and since this is uniform it is also the instantaneous speed. The displacement is $r\sqrt{2} = 42.42$ ft. at an angle of 45° with the diameter through the starting-point. The mean velocity is then 4.242 ft. per sec. in the same direction. The instantaneous velocity is at any instant tangent to the circle at the point at that instant, and equal in magnitude to the instantaneous speed, or 4.712 ft. per sec.

(8) *A man walks at the rate of* 4 *miles per hour in a rain-storm, and the drops fall vertically with a speed of* 200 *ft. per sec. In what direction will they seem to him to fall?*

Ans. Inclined 1° 40′.8 to the vertical.

(9) *A ship sails east with a speed of* 12 *miles an hour, and a shot is fired so as to strike an object which bears N.E. If the gun gives the shot a mean horizontal velocity of* 90 *ft. per sec., towards what point of the compass must it point?*

Ans. N. 37° 3′ E.

(10) *A man* 6 *ft. tall walks at the rate of* 4 *miles per hour directly away from a lamp-post* 10 *ft. high. Find the velocity of the extremity of his shadow.*

Ans. 10 miles per hour in the direction he is walking (see Ex. 14, page 19).

(11) *Two points moving with uniform speed v start at the same instant in the same direction from the point of contact of their paths. The one moves in a circle of radius r, the other in a tangent to the circle. Find their relative velocity at the end of the time t.*

Ans. $2v \sin \frac{vt}{2r}$ radians in a direction inclined to the tangent at an angle $\frac{1}{2}\left(\pi - \frac{vt}{r}\right)$ radians.

(12) *A moves N.E. with a velocity v, and B moves S.* 15° *E. with the same velocity. Find A's velocity relative to B.*

Ans. $v\sqrt{3}$, direction N. 15° E.

(13) *A railway train runs* 30 *miles per hour north. Another running* 15 *miles per hour appears to a passenger in the first to be running at* 25 *miles per hour. Find the direction of the velocity of the latter.*

Ans. N. 56° 15′ E. or N. 56° 15′ W.

(14) *Two candles A and B, each* 1 *ft. long and requiring* 4 *and* 6 *hours respectively to burn out, stand vertically at a distance of* 1 *ft. The shadow of B falls on a vertical wall at a distance of* 10 *ft. from B. Find the speed of the end of the shadow.*

Ans. 8 inches per hour.

(15) *A ship has a northeasterly velocity of* 12 *knots per hour Find the magnitude of her velocity* (*a*) *in an easterly direction;* (*b*) *in a direction* 15° *W. of N.*

Ans. (*a*) $6\sqrt{2}$; (*b*) 6 knots per hour.

(16) *A boat-crew row* 3½ *miles down a river and back again in* 1 *hour* 40 *minutes. If the river has a current of* 2 *miles per hour, find the rate at which the crew would row in still water.*

Ans. 5 miles per hour.

(17) *A steamer goes* 9.6 *miles per hour in still water. How long will it take to run* 10 *miles up a stream and return, the velocity of the stream being* 2 *miles an hour?*

Ans. 2 hours 11 minutes.

(18) *A steam tug travels* 10 *miles an hour in still water, but draws a barge* 4 *miles an hour. It has to take the barge* 10 *miles up a stream which runs* 1 *mile an hour, and then return without the barge. How long will it take for the job?*

Ans. $4\frac{8}{33}$ hours.

(19) *A vessel makes two runs on a measured mile, one with the tide in a minutes and one against the tide in b minutes. Find the*

velocity of the vessel through the water, and of the tide, supposing both uniform.

Ans. $30\frac{a+b}{ab}$ and $30\frac{b-a}{ab}$ miles per hour.

(20) *A point receives simultaneously three velocities,* 60 *ft. per sec. N.,* 88 *ft. per sec. W.* 30° *S., and* 60 *ft. per sec. E.* 30° *S. Give the magnitude and direction of the resultant velocity.*

Ans. 28 feet W. 30° S.

(21) *A ship sailing due north at the rate of* 8 *miles per hour is carried to the east by a current of* 4 *miles per hour. Find the velocity with reference to the land.*

Ans. 8.94 miles per hour N. 26° 34′ E.

(22) *A ship is sailing E.* 22½° *S. at the rate of* 10 *miles an hour and the wind seems to blow from the N.W. with a velocity of* 6 *miles an hour. Find the actual velocity of the wind.*

Ans. 15.7 miles an hour W. 30° 55′ N.

(23) *A point moves in t seconds from A to B, the positions being given by the co-ordinates x_1, y_1 and x_2, y_2. What is the mean velocity?*

Ans. $v = \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{t}$, making an angle α with the axis of x given by $\tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$.

(24) *A point has four component velocities in the same plane, of* 12, 24, 36, 48 *ft. per sec., making with the axis of X the angles of* 16°, 29°, 33°, 75° *respectively. What is the resultant velocity?*

Ans. $v_x = 75.14$; $v_y = 80.915$; $v_r = 110.424$; angle with axis of x, $a = 47°\ 7'\ 10''$; angle with axis of y, $b = 42°\ 52'\ 50''$.

(25) *A point has three component velocities in the same plane given by $v_1 = 40$, $v_2 = 50$, $v_3 = 60$ ft. per sec., making with the axes of X and Y angles given by $\alpha_1 = 60°$, β_1 obtuse; $\beta_2 = 30°$, α_2 obtuse; $\alpha_3 = 120°$, β_3 obtuse. Find the resultant velocity.*

Ans. We have $\beta_1 = 150°$, $\alpha_2 = 120°$, $\beta_3 = 150°$. Hence $v_x = -35$ ft. per sec., $v_y = -43.3$ ft. per sec., $v_r = 55.67$ ft. per sec., making the angles with X and Y given by $a = 128°\ 57'\ 17''$, $b = 141°\ 2'\ 43''$.

(26) *A point has the component velocities $v_1 = 40$, $v_2 = 50$, $v_3 = 60$ ft. per sec., making the angles with X, Y, Z, $\alpha_1 = 60°$, $\beta_1 = 100°$, γ_1 obtuse; $\alpha_2 = 100°$, $\beta_2 = 60°$, γ_2 acute; $\alpha_3 = 120°$, $\beta_3 = 100°$, γ_3 acute. Find the resultant velocity.*

Ans. We find the angles γ (page 12) by the equation

$$\cos^2\gamma = -\cos(\alpha + \beta)\cos(\alpha - \beta).$$

Hence $\gamma_1 = 148°\ 2'\ 31''.7$, $\gamma_2 = 31°\ 57'\ 28''.3$, $\gamma_3 = 31°\ 57'\ 28''.3$.

$v_x = -18.6824$ ft. per sec., $v_y = +7.635$ ft. per sec., $v_z = +59.391$ ft. per sec.,

$$v_r = 62.73 \text{ ft. per sec.},$$

making with the axes of X, Y, Z, angles given by

$$a = 107°\ 19'\ 36'',\quad b = 38°\ 0'\ 33'',\quad c = 18°\ 46'\ 42''.$$

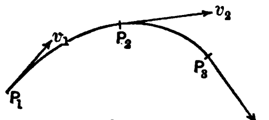
CHAPTER V.

ACCELERATION. RESOLUTION AND COMPOSITION OF ACCELERATIONS.

ANALYTICAL DETERMINATION OF RESULTANT FOR CONCURRING ACCELERATIONS. EQUATIONS OF MOTION. THE HODOGRAPH. TANGENTIAL AND NORMAL ACCELERATION.

Mean Acceleration.—Let P_1, P_2, etc., Fig. (*a*), be the path of a moving point P, and let the corresponding instantaneous velocities be v_1, v_2, etc.

Fig. (*a*).



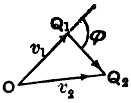


Fig. (*b*).

Each velocity is tangent to the path at the corresponding point and is equal in magnitude to the speed at that point.

If t is the number of units of time in passing from P_1 to P_2, the mean speed for that time (page 15) is $\frac{\textit{path } P_1P_2}{t}$ *units of speed*. The integral change of speed in the time t is $v_2 - v_1$ (page 24), and the mean rate of change of speed in the time t is $a = \frac{v_2 - v_1}{t}$ (page 24).

If now in Fig. (*b*) we draw OQ_1 parallel and equal in magnitude to v_1 and OQ_2 parallel and equal to v_2, then the integral change of speed is represented by $OQ_2 - OQ_1$ and the mean rate of change of speed by $\frac{OQ_2 - OQ_1}{t}$.

The *change of velocity*, however, in the time t is represented in magnitude and direction by Q_1Q_2, and this we call the **integral acceleration.**

The mean time-rate of change of velocity is given in magnitude by $\frac{Q_1Q_2}{t}$, and in direction by Q_1Q_2. We call this the **mean linear acceleration.** The term acceleration always means linear acceleration unless otherwise specified.

Mean acceleration is then time-rate of change of velocity, whether that change takes place in the direction of motion or not.

Instantaneous Acceleration.—The limiting magnitude and direction of the mean acceleration, when the interval of time is indefinitely small, is the **instantaneous acceleration.**

The limiting direction is not necessarily tangent to the path except in the case of rectilinear motion, and the limiting magnitude is not the rate of change of speed in the path, except in the case of rectilinear motion.

The term acceleration always signifies instantaneous acceleration unless otherwise specified. It is the limiting time-rate of change of velocity, whether that change take place in the direction of the motion or not.

Acceleration may be zero, uniform or variable. When it is zero the velocity is uniform, and we have uniform speed in a straight line.

When it is uniform, it has the same magnitude and the same unchanged direction, whatever the interval of time. In such case the acceleration at any instant is equal to the mean acceleration for any interval of time. If its direction coincides with that of the velocity, we have uniform rate of change of speed in a straight line. If it does not so coincide, we have uniform acceleration and motion in a curved line.

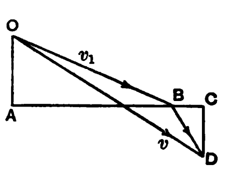
When it is variable, either direction or magnitude changes, or both change.

Unit of Acceleration.—The magnitude of the unit of acceleration is evidently the same as that for rate of change of speed, viz., one unit of length-per-sec. per sec., as for instance one foot-per-sec. per sec. We denote the magnitude of the acceleration thus measured by the letter f, to distinguish it from rate of change of speed, which we have denoted by a (page 25).

Line Representative of Acceleration.—Since acceleration is time-rate of change of velocity, and is therefore, like velocity and displacement, a vector quantity, it can be represented like them by a straight line, whose length and direction give the magnitude and direction of the acceleration (pages 34, 43). Rate of change of speed is given by stating sign and magnitude only. It is a scalar quantity (page 25).

Triangle and Polygon of Accelerations.—The principles, therefore, of page 35 hold good for accelerations also, as well as displacements. We have then the "triangle and polygon of accelerations."

Composition and Resolution of Accelerations.—For the same reason we can combine and resolve accelerations in the same way as displacements, and the principles of pages 35, 36 apply.



Let OB and OD be the initial and final velocities of a point in any given time t.

Then BD is the integral acceleration and $\frac{BD}{t}$ is the mean acceleration, or the instantaneous acceleration if it is uniform. If not uniform, $\frac{BD}{t}$, where t is indefinitely small, gives the acceleration.

Draw OA and DC at right angles to any line AC through B in any given direction.

Then $\frac{BC}{t}$ is the component of the acceleration in this direction.

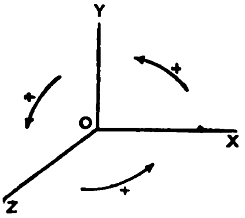
But $\frac{BC}{t} = \frac{AC - AB}{t}$, and AC and AB are the components of the velocities in the direction AC.

Hence *the component in any direction of an acceleration is equal to the acceleration of the component velocities in that direction.*

Sign of Components of Acceleration.—We have the same rule for the signs of the horizontal and vertical components f_x and f_y of any acceleration f, as for the horizontal and vertical components of v_x and v_y of any velocity v (page 44).

If the direction of the linear representative is towards the right or upwards, f_x and f_y are positive; if towards the left or downward, f_x and f_y are negative.

The sign, then, applied to component accelerations indicates direction of action. For rectangular co-ordinates (+) signifies in the direction OX, OY, OZ, (−) in the opposite directions.



If polar co-ordinates are used, the component acceleration along the radius vector is positive (+) when it acts away from the pole, and negative (−) when it acts towards the pole.

Evidently, then, we measure angles in the plane XY from OX around towards OY, in the plane YZ from OY around towards OZ, in the plane ZX from OZ around toward OX, as shown by the arrows in the figure.

Take, for instance, the case of a particle projected vertically away from the earth with the initial velocity v_1 and attaining the final velocity v.

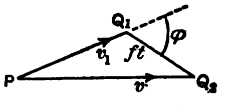
As long as the particle ascends, the direction of v is upwards, and v, v_1 are both positive. The acceleration due to gravity is always downwards, and hence is negative.

When the particle is descending with the initial velocity v_1, then both v_1 and v are negative, and the acceleration is negative as before.

Analytical Determination of the Resultant for Concurring Accelerations.—When the line representatives meet in a point, they are said to be *concurring*. We have then the same expressions for the magnitude and direction of the resultant of any number of concurring component accelerations as for displacements (pages 37 and 38). We have only to substitute f in place of d.

Equations of Motion of a Point under Different Accelerations.—The magnitude of the acceleration in general is not the same as rate of change of speed, except in the case of rectilinear motion. We have therefore denoted it by f, to distinguish it from rate of change of speed, which we have denoted by a. If, then, we denote by f_t the magnitude of the tangential acceleration, or the tangential component of f, we have $f_t = a$, or the magnitude *of the tangential acceleration* is equal to the rate of change of speed.

(*a*) **Acceleration Zero.**—We have in this case the line representative $Q_1Q_2 = 0$, and hence the line representative of the velocity does not change either in direction or magnitude.



We have then rectilinear motion with constant velocity, and for any interval of time

$$v = \frac{s - s_1}{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where s_1 and s are the initial and final distances of the point from any fixed point in the line. This equation is general if we pay attention to the sign of displacement and velocity (pages 37 and 44).

(*b*) **When the Direction of the Acceleration coincides with the Direction of the Velocity.**—In this case the line representative Q_1Q_2

coincides in direction with $PQ_1 = v_1$. We have then rectilinear motion with varying velocity.

If f is uniform, the instantaneous acceleration is equal to the mean acceleration for any interval of time.

Hence, if v_1 is the initial and v the final velocity, we have *for f constant*

$$f = \frac{v - v_1}{t}. \quad \text{(2)}$$

This value of f is general when we pay attention to the sign for velocity and acceleration (pages 44 and 50).

From (2) we have

$$v = v_1 + ft. \quad \text{(3)}$$

The average speed is

$$\frac{v + v_1}{2} = v_1 + \frac{1}{2}ft. \quad \text{(4)}$$

The displacement is

$$s - s_1 = \frac{v + v_1}{2}t = v_1t + \frac{1}{2}ft^2. \quad \text{(5)}$$

Inserting the value of t from (2) we have

$$s - s_1 = \frac{v^2 - v_1^2}{2f}. \quad \text{(6)}$$

Hence $$v^2 = v_1^2 + 2f(s - s_1). \quad \text{(7)}$$

[*If f is variable,* we have from (1), in Calculus notation,

$$v = \frac{ds}{dt}; \quad \text{(8)}$$

and from (2),

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2}; \quad \text{(9)}$$

and from (8),

$$s - s_1 = \int_{t=0}^{t=t} vdt. \quad \text{(10)}$$

The preceding equations can be deduced from these as on page 28.]

All these equations are precisely similar to those on page 28, except that we have f in place of a.

(*c*) **When the Acceleration is Constant in Magnitude and Direction, but makes an Angle ϕ with the Initial Velocity.**—In this case we have motion in a curve, and $Q_1Q_2 = ft$. Hence

$$v^2 = (v_1 + ft\cos\phi)^2 + (ft\sin\phi)^2 = v_1^2 + 2v_1ft\cos\phi + f^2t^2. \quad \text{(11)}$$

The tangent of the angle Q_1PQ_2 is

$$\tan Q_1PQ_2 = \frac{ft\sin\phi}{v_1 + ft\cos\phi}. \quad \text{(12)}$$

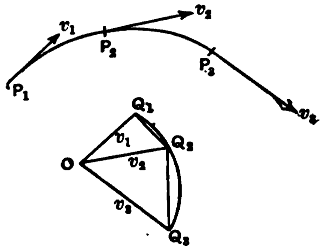
The displacement in the time t is given by

$$d^2 = \left(v_1t + \frac{1}{2}ft^2\cos\phi\right)^2 + \left(\frac{1}{2}ft^2\sin\phi\right)^2; \quad \text{(13)}$$

and the tangent of its angle of inclination to v_1 is

$$\frac{\frac{1}{2}ft^2\sin\phi}{v_1t + \frac{1}{2}ft^2\cos\phi}. \quad \text{(14)}$$

The Hodograph.—Let a point P moving in any curve have the positions P_1, P_2, P_3, etc., and let the magnitude of the corresponding velocities be v_1, v_2, v_3, etc. These velocities are tangent to the path at P_1, P_2, P_3, and are equal in magnitude to the speed at these points.



If from a point O we draw lines OQ_1, OQ_2, OQ_3, etc., parallel and equal to v_1, v_2, v_3, the extremities of these lines will form a polygon.

If, however, the points P_1, P_2, P_3 are indefinitely near, the polygon becomes a curve $Q_1Q_2Q_3$, such that when the point P describes the path $P_1P_2P_3$, we can conceive another point Q to describe the curve $Q_1Q_2Q_3$. This curve is called the **hodograph.*** The point O is called the *pole* of the hodograph. The points Q_1, Q_2, Q_3 are called the *points corresponding* to P_1, P_2, P_3.

Any radius vector, as OQ_1, OQ_2, in the hodograph, represents in magnitude and direction the velocity at the corresponding point P_1, P_2, etc., of the path. The magnitude of OQ_1, OQ_2, etc., is the speed v_1, v_2, etc., at P_1, P_2, etc. The direction of OQ_1, OQ_2 is the direction of v_1, v_2, or that of the tangent to the path at P_1, P_2.

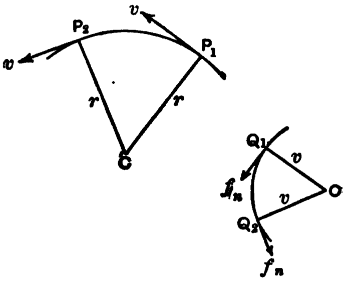
If t is the interval of time in moving from P_1 to P_2, then the chord Q_1Q_2 in the hodograph gives the magnitude and direction of the integral acceleration for that time, and $\frac{\text{chord } Q_1Q_2}{t}$, which is the mean velocity in the hodograph, gives the mean acceleration in the path.

When the time is indefinitely small, Q_1Q_2 becomes tangent to the hodograph and $\frac{Q_1Q_2}{t}$, which is now the instantaneous velocity in the hodograph, gives the instantaneous acceleration in the path.

The tangent to the hodograph at any point, therefore, gives the direction of the instantaneous acceleration at the corresponding point of the path. The instantaneous speed in the hodograph gives the magnitude of this acceleration.

Hence, *the velocity at any point of the hodograph is the acceleration at the corresponding point of the path.*

Tangential and Normal Acceleration.—The entire resultant acceleration f at any point of the path may be resolved into a component tangential to the path f_t and a component normal to the path f_n, so that $f = \sqrt{f_t^2 + f_n^2}$.



The magnitude of the tangential component f_t is the rate of change of speed in the path. Its direction is always tangent to the path, or parallel to the radius vector of the hodograph at the corresponding point of the hodograph.

In order to find the normal component f_n, let us first suppose a point to move in a circle with constant speed v.

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* Invented by Sir W. R. Hamilton.

Let the radius of the circle be r, and take any two points P_1 and P_2 on the circle. Then the velocity at P_1 is v tangent at P_1 or perpendicular to r at that point, and the velocity at P_2 is v tangent at P_2 or perpendicular to r at P_2.

Now construct the hodograph by making OQ_1 parallel and equal to the velocity at P_1 and OQ_2 parallel and equal to the velocity at P_2, etc.

Evidently the hodograph is also a circle of radius v, and the speed in the hodograph is also constant, since the point P moves with constant speed and makes a revolution in the same time as its corresponding point Q in the hodograph. Let t be the time of revolution; then $\frac{2\pi v}{t}$ is the speed of Q in the hodograph or the acceleration of P in the path; and since this speed at any point Q_1 is at right angles to OQ_1 or v, it is normal to the path at P_1 or parallel to CP_1. We have then $f_n = \frac{2\pi v}{t}$. But the speed in the path is $v = \frac{2\pi r}{t}$. Hence $t = \frac{2\pi r}{v}$, and substituting, we have

$$f_n = \frac{v^2}{r}.$$

We can obtain the same result as follows: Since the point P moves from P_1 to P_2 in the same time that Q moves from Q_1 to Q_2, and the angle P_1CP_2 is equal to the angle Q_1OQ_2, we have

$$f_n : v :: v : r, \quad \text{or} \quad f_n = \frac{v^2}{r}.$$

Since we have supposed v constant in magnitude, the tangential acceleration f_t is zero, and therefore f_n in this case is the magnitude of the total resultant acceleration f.

A normal acceleration, then, has no effect upon the speed, but only changes the direction of motion.

Let us now suppose that the speed v is not constant, the point still moving in a circle. Then the hodograph will not be a circle.

But if the two points P_1 and P_2 are indefinitely near, so that the arc P_1P_2 is indefinitely small, the velocities at P_1 and P can be taken as equal still, and we shall still have

$$f_n : v :: v : r, \quad \text{or} \quad f_n = \frac{v^2}{r}.$$

Again, if the point P moves in any curve whatever, a circle can always be described whose curvature is the same as that of the curve at any given point. The radius of this circle is the radius of curvature ρ at the point.

In this case, then, we should have

$$f_n : v :: v : \rho, \quad \text{or} \quad f_n = \frac{v^2}{\rho}.$$

Therefore, in general, whatever the path may be and whatever the speed in the path—

The magnitude at any instant of the normal component of the acceleration is equal to the square of the speed at that instant divided by the radius of curvature.

EXAMPLES.

(1) *A point moves with uniform velocity in a straight line. What is the hodograph?*

Ans. A point.

(2) *A point moves with uniform acceleration either in a straight line or a curve. What is the hodograph? What is the speed in the hodograph?*

Ans. A straight line. Uniform.

(3) *Show that the direction of motion of any point B on the circumference of a circle rolling with velocity v on a straight line is perpendicular to AB at any instant, when A is the point of contact of the circle with the straight line at the instant considered.*

(4) *AB is a diameter of a circle of which BC is a chord. When is the moment about A of a velocity represented by BC the greatest?*

Ans. When BC and AC are equal.

(5) *A point is moving with uniform speed of a mile in 2 min. 40 sec. round a ring of 100 ft. radius. Find the acceleration.*

Ans. 10.89 ft.-per-sec. per sec. towards the centre.

(6) *A point moves in a horizontal circle with uniform speed v, starting from the north point and moving eastward. Find the integral acceleration when it has moved (a) through a quadrant; (b) a semicircle; (c) three quadrants.*

Ans. (*a*) $v\sqrt{2}$, SW.; (*b*) $2v$, W.; (*c*) $v\sqrt{2}$, NW.

(7) *If the component velocities of a moving point are represented by the sides of a plane polygon, taken the same way round, the algebraic sum of their moments about any point in their plane is zero.*

Ans. If the polygon closes, the resultant velocity is zero. If it does not close, the line necessary to make it close taken the other way round is the resultant. In either case the algebraic sum of the moments is zero.

(8) *Show that the hodograph of a point moving with uniform speed in a circle is a circle in which the corresponding point moves also with uniform speed.*

(9) *Show that the locus of the extremity of a straight line representing either of the two equal components of a given acceleration is a straight line perpendicular to the straight line representing the given acceleration and through its middle point.*

(10) *Let the velocity of a point moving in a straight line vary as the square root of the product of its distances from two fixed points in the line. Show that its instantaneous acceleration varies as the mean of its distances from the fixed points.*

Ans. Let s be one distance and $a + s$ the other. Then from page 51, Chap. V, $v = \frac{ds}{dt} = k\sqrt{s(a+s)}$, where k is a constant.

$$\text{Then } a = \frac{dv}{dt} = \frac{kads + 2ksds}{2dt\sqrt{as + s^2}} = \frac{k^2a + 2k^2s}{2} = \frac{k^2(a + 2s)}{2}.$$

$$\text{But the mean of the distances is } \frac{s + (a + s)}{2} = \frac{a + 2s}{2}.$$

(11) *If the algebraic sum of the moments of the component velocities of a moving point about any two points P and Q are each zero, show that the algebraic sum of their moments about any point in the line PQ will be zero.*

(12) *A moving point P has two component velocities one of which is double the other. The moment of the smaller about a point O in the plane is double that of the greater. Find the magnitude and direction of the resultant velocity.*

Ans. If a is the smaller component and α, β are the inclinations of the greater and smaller components respectively to PO, the resultant is $a\sqrt{5+4\cos(\beta+\alpha)}$, and it is inclined to PO at an angle whose sin is

$$\frac{2\sin\alpha+\sin\beta}{\sqrt{5+4\cos(\beta+\alpha)}}.$$

(13) *The velocity v of a point moving in a straight line varies as the square root of its distance s from a fixed point in the line. What is its instantaneous acceleration?*

Ans. We have $v=\dfrac{ds}{dt}=k\sqrt{s}$, where k is a constant. Hence

$$a=\frac{dv}{dt}=\frac{kds}{2dt\sqrt{s}}=\frac{kv}{2\sqrt{s}}=\frac{v^2}{2s}=\frac{k^2}{2}.$$

(14) *Two railway trains move in directions inclined* 60°. *The one, A, is increasing its speed at the rate of* 4 *ft.-per-min. per min. The other, B, has the brakes on and is losing speed at the rate of* 8 *ft.-per-min. per min. Find the relative acceleration.*

Ans. $4\sqrt{7}$ ft. per min. per min., inclined to the direction of A at an angle whose sin is $\sqrt{\frac{3}{7}}$ and to the direction of B at an angle whose sin is $\frac{1}{2}\sqrt{\frac{3}{7}}$.

(15) *The initial and final velocities of a moving point during an interval of two hours are* 8 *miles per hour E.* 30° *N. and* 4 *miles per hour N. Find the integral and the mean acceleration.*

Ans. $4\sqrt{3}$ miles per hour W., $2\sqrt{3}$ miles-per-hour per hour W.

(16) *A point moves in a circle of radius* 8 *inches and has at a given position a speed of* 4 *in. per sec., which is changing at the rate of* 6 *in.-per-sec. per sec. Find* (*a*) *the tangential acceleration;* (*b*) *the normal acceleration;* (*c*) *the resultant acceleration.*

Ans. (*a*) 6 in.-per-sec. per sec.; (*b*) 2 in.-per-sec. per sec.; (*c*) $2\sqrt{10}$ in.-per-sec. per sec.

(17) *Newton assumed that the acceleration of gravity varied inversely as the square of the distance from the earth's centre. He then tested this assumption by applying it to the moon. Assuming the acceleration at the earth's surface* 32.2 *ft.-per-sec. per sec., the radius of the earth* 4000 *miles, the distance between centres of earth and moon* 240,000 *miles, and the speed of the moon in its orbit around the earth* 3375 *ft. per sec., show that Newton's assumption is in accord with fact.*

Ans. The acceleration of the moon's centre towards the earth is $\dfrac{v^2}{r}$, or $\dfrac{(3375)^2}{240000\times 5280}=0.0089$ ft.-per-sec. per sec.

But according to Newton's assumption, if g' is the acceleration at the distance of the moon, $\dfrac{g'}{32.2}=\dfrac{(4000)^2}{(240000)^2}=\dfrac{1}{3600}$. Hence $g'=0.0089$ ft.-per-sec. per sec.

(18) *Find the resultant of four component accelerations represented by lines drawn from any point P within a parallelogram to the angular points.*

Ans. If O is the intersection of the diagonals, PO represents the direction of the resultant, and $4PO$ its magnitude.

(19) *A ball is let fall in an elevator which is rising with an acceleration of* 7.2 *kilometers-per-min. per min. The acceleration of the ball relative to the earth is* 981 *cm.-per-sec. per sec. Find its acceleration relative to the elevator.*

Ans. 1181 cm.-per-sec. per sec. towards the floor.

(20) *Assuming the mean radius of the earth* 6370900 *meters, the speed of a point on the equator* 465.1 *m. per sec., acceleration of a falling body* 9.81 *m.-per-sec. per sec., find with what velocity a shot must be fired at the equator with either a westerly or easterly direction in order that, if unresisted, it shall move horizontally round the earth, completing its circuit in* $1\frac{1}{3}$ *or* $1\frac{1}{2}$ *hours respectively.*

Ans. Westerly velocity, 8370.7 meters per sec.; easterly velocity, 7440.5 meters per sec.

(21) *If different points describe different circles with uniform speeds and with accelerations proportional to the radii of their paths, show that their periodic times will be the same.*

(22) *The resultant of two accelerations a and a′ at right angles to one another is R. If a is increased by* 9 *units and a′ by* 5 *units, the magnitude of R becomes three times its former value, and its direction becomes inclined to a at the angle of its former inclination to a′. Find a, a′ and R.*

Ans. 3, 4 and 5 units respectively.

(23) *If a tangent be drawn at any point of a conic section, the locus of the foot of the perpendicular let fall from a focus on this tangent is a circle in the case of the ellipse and hyperbola, and a straight line in the case of a parabola. Also the locus of the foot of a perpendicular from the vertex of a parabola on a straight line drawn through the focus is a circle.*

Assuming these properties, show that if a point move in either a circle, ellipse, hyperbola or parabola, so that the moment of its velocity about a focus is constant, the hodograph is a circle.

(24) *Show that if a point moves in an ellipse so that the moment of its velocity about the centre is constant, the hodograph is an ellipse.* [*Note that the area of the parallelogram formed by drawing tangents to an ellipse at the extremities of a pair of conjugate diameters is constant.*]

(25) *A bullet is fired in a direction towards a second bullet which is let fall at the same instant. Show that the line joining them will move parallel to itself and that the bullets will meet.*

(26) *Determine whether any of the following equations are possible or not:*

$$(1)\ 10avst + 8v^2s = 3g^3t^4\,;$$
$$(2)\ v^3t - 4as + 3a = 0\,;$$
$$(3)\ 6v + 2g^3as^2t^8 = 3a^2st^4.$$

Ans. The first gives us $\frac{[L]^3}{[T]^2}$ in each term and is therefore possible. The second gives us $\frac{[L]^3}{[T]^2}$, $\frac{[L]^2}{[T]^2}$ and $\frac{[L]}{[T]^2}$, or each term refers to different kinds of quantities, and the equation is nonsense on its face. The third gives us $\frac{[L]}{[T]}$, $[L]^6$ and $[L]^3$, and is also nonsense.

(27) *A point moves in a straight line so that the number of units of distance s from the origin at the end of any number of seconds t is given by $s = 2 + \frac{5}{2}t + \frac{3}{4}t^2 + \frac{5}{8}t^3$. Find* (*a*) *the number of units of distance from the origin at the start;* (*b*) *the velocity v at any instant;* (*c*) *the acceleration a at any instant;* (*d*) *the velocity at the start;* (*e*) *the acceleration at the start.*

Ans. (*a*) 2 units of distance; (*b*) $v = \frac{5}{2} + \frac{3}{2}t + \frac{15}{8}t^2$; (*c*) $a = \frac{3}{2} + \frac{15}{4}t$; (*d*) $\frac{5}{2}$ units of distance per sec.; (*e*) $\frac{3}{2}$ units of distance-per-sec. per sec.

(28) *A point moves in a straight line so that the number of units of acceleration a at the end of any number of seconds t is given by $a = 7 - \frac{1}{3}t + 2t^2$. If v_1 is the number of units of velocity at the start, and s_1 the number of units of distance from the origin at the start, find the velocity and the distance from the origin at any instant.*

Ans. $v = v_1 + 7t - \frac{1}{6}t^2 + \frac{2}{3}t^3$;

$$s = s_1 + v_1t + \frac{7}{2}t^2 - \frac{1}{18}t^3 + \frac{1}{6}t^4.$$

(29) *A point moves in a straight line so that the number of units of velocity v at the end of any number of seconds t is given by $v = 5 - \frac{3}{2}t + \frac{5}{6}t^2$. Find the acceleration a and the distance s, if s_1 is the initial distance.*

Ans. $a = -\frac{3}{2} + \frac{5}{3}t$;

$$s = s_1 + 5t - \frac{3}{4}t^2 + \frac{5}{18}t^3.$$

(30) *A point has three component accelerations in the same plane given by $f_1 = 40$, $f_2 = 50$, $f_3 = 60$ ft.-per-sec. per sec., making with the axes of X and Y angles given by $\alpha_1 = 60°$, β_1 obtuse; $\beta_2 = 30°$, α_2 obtuse; $\alpha_3 = 120°$, β_3 obtuse. Find the resultant acceleration.*

Ans. We have $\beta_1 = 150°$, $\alpha_2 = 120°$, $\beta_3 = 150°$. Hence

$f_x = -35$ ft.-per-sec. per sec.; $f_y = -43.3$ ft.-per-sec. per sec., and
$f_r = 55.67$ ft.-per-sec. per sec.,

making the angles with X and Y given by

$$a = 128° \, 57' \, 17'', \quad b = 141° \, 2' \, 43''.$$

(31) *A point has three component accelerations, $f_1 = 40$, $f_2 = 50$, $f_3 = 60$ ft.-per-sec. per sec., making with the axes of X, Y, Z angles given by $\alpha_1 = 60°$, $\beta_1 = 100°$, γ_1 obtuse; $\alpha_2 = 100°$, $\beta_2 = 60°$, γ_2 acute; $\alpha_3 = 120°$, $\beta_3 = 100°$, γ_3 acute. Find the resultant acceleration.*

Ans. We find the angles γ (page 12) by the equation

$$\cos^2 \gamma = -\cos(\alpha + \beta)\cos(\alpha - \beta).$$

Hence $\gamma_1 = 148° \; 2' \; 31''.7$, $\gamma_2 = 81° \; 57' \; 28''.3$, $\gamma_3 = 31° \; 57' \; 28''.3$;

$f_x = -18.6824$ ft.-per-sec. per sec., $f_y = +7.635$ ft.-per-sec. per sec.,

$f_z = +59.391$ ft.-per-sec. per sec., $f_r = 62.73$ ft.-per-sec. per sec.,

making with the axes of X, Y, Z angles given by

$$a = 07° \; 19' \; 36'', \quad b = 83° \; 0' \; 33'', \quad c = 18° \; 46' \; 42''.$$

(32) *Investigate the motion of a point whose initial velocity in a horizontal direction is* 0, *and in a vertical direction* -32 *ft. per sec. The horizontal acceleration is* $f_x = +16$ *ft.-per-sec. per sec., and the vertical acceleration* $f_y = +4t$ *ft.-per-sec. per sec.*

Ans. The resultant acceleration is (page 51)

$$f = \sqrt{(16)^2 + (4t)^2},$$

and it makes an angle λ with the horizontal whose cosine is

$$\cos \lambda = \frac{16}{\sqrt{(16)^2 + (4t)^2}},$$

and an angle μ with the vertical whose cosine is

$$\cos \mu = \frac{4t}{\sqrt{(16)^2 + (4t)^2}}.$$

The horizontal velocity at the end of any time is

$$v_x = 16t.$$

The vertical velocity at the end of any time is

$$v_y = -32 + 2t^2.$$

The resultant velocity is

$$v = \sqrt{(16t)^2 + (2t^2 - 32)^2} = 2t^2 + 32,$$

and it makes an angle α with the horizontal whose cosine is

$$\cos \alpha = \frac{8t}{t^2 + 16},$$

and an angle β with the vertical whose cosine is

$$\cos \beta = \frac{t^2 - 16}{t^2 + 16}.$$

The distance s described in any time is

$$s = \frac{2}{3}t^3 + 32t.$$

The tangential acceleration is

$$f_t = a = f_x \cos \alpha + f_y \cos \beta = 16 \times \frac{8t}{t^2 + 16} + 4t \times \frac{t^2 - 16}{t^2 + 16} = 4t.$$

The normal acceleration is (page 52)

$$f_n = \sqrt{f^2 - f_t^2} = 16.$$

The radius of curvature is (page 53)

$$\rho = \frac{v^2}{f_n} = \frac{(2t^2 + 32)^2}{16}.$$

The horizontal distance described is

$$x = 8t^2.$$

The vertical distance described is

$$y = \frac{2}{3}t^3 - 32t.$$

Eliminating t, we have for the equation of the path

$$y = \sqrt{\frac{x}{8}}\left(\frac{x}{12} - 32\right).$$

(33) *Investigate the motion of a point whose initial velocity in the direction of the axis of x is $+ 2$ ft. per sec.; in the direction of the axis of y, 0; in the direction of the axis of z, $+ 4$ ft. per sec. The acceleration in the direction of the axis of x is $f_x = 0$; in the direction of the axis of y, $f_y = + 3$ ft.-per-sec. per sec.; in the direction of the axis of z, $f_z = + 5$ ft.-per-sec. per sec.*

Ans. The resultant acceleration is (page 51)

$$f = \sqrt{0^2 + 3^2 + 5^2} = \sqrt{34},$$

which makes an angle λ with the axis of x whose cosine is

$$\cos\lambda = \frac{0}{\sqrt{34}} = 0;$$

with the axis of y an angle μ whose cosine is

$$\cos\mu = \frac{3}{\sqrt{34}};$$

with the axis of z an angle ν whose cosine is

$$\cos\nu = \frac{5}{\sqrt{34}}.$$

The velocities in the direction of x, y and z are

$$v_x = 2; \quad v_y = 3t; \quad v_z = 4 + 5t.$$

The resultant velocity is

$$v = \sqrt{2^2 + (3t)^2 + (4 + 5t)^2} = \sqrt{34t^2 + 40t + 20},$$

which makes angles α, β and γ with the axes of x, y and z given by

$$\cos\alpha = \frac{2}{\sqrt{34t^2 + 40t + 20}}; \qquad \cos\beta = \frac{3t}{\sqrt{34t^2 + 40t + 20}};$$

$$\cos\gamma = \frac{4 + 5t}{\sqrt{34t^2 + 40t + 20}}.$$

The distances described in the direction of x, y and z are

$$x = 2t; \quad y = \frac{3}{2}t^2; \quad z = 4t + \frac{5}{2}t^2.$$

If we eliminate t, we have

$$y = \frac{3}{8}x^2;$$

$$z = 2x + \frac{5}{8}x^2.$$

These are the equations of the projection of the path upon the planes of xy and xz.

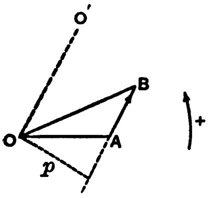
CHAPTER VI.

MOMENT OF DISPLACEMENT, VELOCITY, ACCELERATION. LINE REPRESENTATIVE.

RESOLUTION AND COMPOSITION OF MOMENTS OF DISPLACEMENT, VELOCITY, ACCELERATION. ANALYTICAL DETERMINATION OF RESULTANT VELOCITY AND MOMENT FOR CONCURRING VELOCITIES, ACCELERATIONS AND DISPLACEMENTS.

Moment of Displacement, Velocity or Acceleration about a Given Point.—The product of the magnitude of a displacement, velocity or acceleration by the magnitude of the perpendicular let fall from any given point upon its direction gives the magnitude of the moment of the displacement, velocity or acceleration with reference to that point.

The perpendicular is called the **lever arm**, and the point is the **centre of moments**.



Thus if AB is the line representative of any displacement, velocity or acceleration, and the perpendicular from any point O is p, the moment relative to O is $\pm AB \times p$. If AB is a displacement d, we have $\pm dp$. If it is a velocity v, we have $\pm vp$. If it is an acceleration f, we have $\pm fp$.

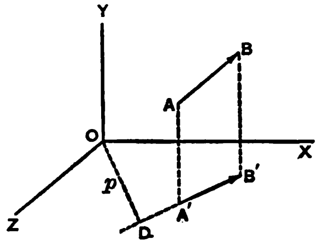
The (+) sign indicates that the radius vector moves around O counter-clockwise as in the figure. The (−) sign denotes clockwise rotation.

If we draw a line OO' through O parallel to AB, it is evident that the moment of AB is the same relative to any point in this line.

Hence, so far as the moment is concerned, a displacement, velocity or acceleration may be considered as laid off from any point in its line of direction.

Moment of Displacement, Velocity or Acceleration about a Given Axis.—Let OY be a given axis and AB any displacement, velocity or acceleration.

Take any plane XZ perpendicular to the axis OY, intersecting this axis in O. Project AB upon this plane in $A'B'$. Now AB can be resolved at A into a component parallel to $A'B'$ and a component perpendicular to $A'B'$ and therefore parallel to the axis OY.

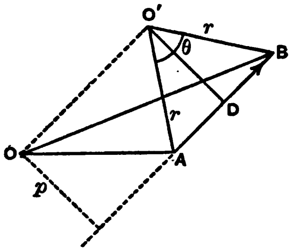


The component parallel to OY causes no rotation about Y. The moment of AB about the axis is then the moment of $A'B'$ about the intersection O of the plane XZ with the axis.

Hence *the moment of a displacement, velocity or acceleration about a given axis is the moment of the component in a plane perpendicular to the axis about the point of intersection of that plane with the axis.*

Here again, so far as moment about any axis is concerned, a displacement, velocity or acceleration may be taken as laid off from any point in its line of direction.

Moment of Displacement.—If $AB = d$ is a displacement, we have then for its moment about any point O, $M = \pm dp$, and dp is twice the area of the triangle AOB.



Hence the moment of a displacement about any point is *twice the area swept through by the radius vector from that point.*

The unit of moment of a displacement is then the square of the unit of length, as *one square foot.*

If we draw OO' parallel to AB and erect a perpendicular DO' at the middle point of AB, then $DO' = p$ and the moment about O is equal to the moment about any point in the line OO'. If $O'A = O'B = r$, and the angle $AO'B = \theta$, then we have for the displacement d

$$d = 2r \sin \frac{\theta}{2}.$$

Moment of Velocity—Acceleration.—If $AB = v$ is a velocity, we have for its moment about any point O, $M = \pm vp$.

Hence the moment of a velocity about any point is *twice the areal velocity of the radius vector.*

The unit of moment of a velocity is then the square of the unit of length per unit of time, as one square foot per second.

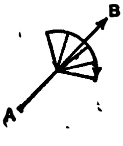
If $AB = f$ is an acceleration, we have for its moment about any point O, $M = \pm fp$.

Hence the moment of acceleration about any point is *twice the areal acceleration of the radius vector.*

The unit of moment of an acceleration is then the square of the unit of length per unit of time squared, or one square ft.-per-sec. per sec.

Line Representative of Moment of Displacement—Velocity—Acceleration.—Since moment of displacement, velocity, acceleration, has thus both magnitude and direction, it is a vector quantity like displacement, velocity, acceleration themselves, and like them can be completely represented by a straight line.

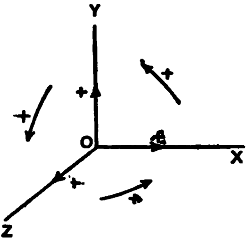
Thus the length of a line represents the magnitude of the moment dp, vp or fp. This line is always taken perpendicular to the plane of rotation of the radius vector, which is therefore known when the direction of the line representative is known. Finally we denote the direction of rotation in the plane by placing an arrowhead on the line, so that *when we look along the line in the direction of this arrow, the rotation is always seen clockwise.*



Thus the line AB denotes by its length the magnitude of the moment dp, vp or fp. The plane of rotation of the radius vector is at right angles to this line, which is therefore coincident with the axis of rotation. The direction of rotation is clockwise in this plane when we look from A to B.

Resolution and Composition of Moments.—The principles, therefore, of pages 35 and 36 hold good for moments of displacements, velocities and accelerations, as well as for displacements, velocities and accelerations themselves. We have then the triangle and polygon of moments.

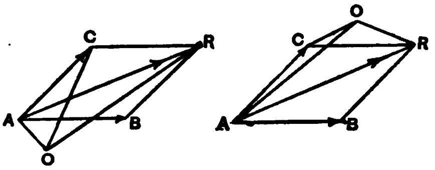
Sign of Components of Moments.—The signs of the line representatives of the components along the axes of *X*, *Y*, *Z* of a moment of displacement, velocity or acceleration follow the same rule as for components of displacement, velocity or acceleration (pages 36, 44, 50).



Hence components in the directions *OX*, *OY*, *OZ* are positive (+), in the opposite direction negative (−). If then we look along the line representatives of the components towards the origin *O*, the rotation is always seen counter-clockwise. Therefore rotation from *X* towards *Y*, *Y* towards *Z*, *Z* towards *X* is positive, in the opposite directions negative.

For polar co-ordinates, directions away from the pole are positive, towards the pole negative.*

The algebraic sum of the moments of any number of component displacements, velocities or accelerations, about any point in their plane, or about any axis, is equal to the moment of the resultant displacement, velocity or acceleration about that point or axis.



Let *AB*, *AC* represent two component displacements, velocities or accelerations of a point *A*. Then the resultant is *AR*. Let *O* be any point in the plane of the components either outside or inside the angle between the resultant and either component.

Then in the first case

$$\text{area } OAR = \text{area } OAB + \text{area } BAR - \text{area } ROB,$$

and in the second case

$$\text{area } OAR = \text{area } OAB + \text{area } ROB - \text{area } BAR.$$

In both cases

$$\text{area } BAR = \text{area } ROB + \text{area } OAC,$$

since all three triangles have equal base *BR*, and the altitude of *BAR* is the sum of the altitudes of *ROB* and *OAC*.

We have then in the first case

$$\text{area } OAR = \text{area } OAB + \text{area } OAC,$$

and in the second case

$$\text{area } OAR = \text{area } OAB - \text{area } OAC.$$

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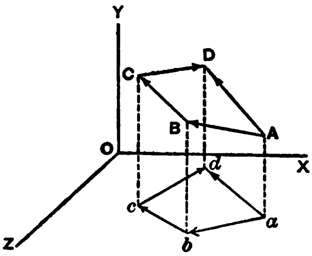
[*] Evidently, then, we measure angles in the plane *XY* from *OX* around towards *OY*; in the plane *YZ* from *OY* around towards *OZ*; in the plane *ZX* from *OZ* around towards *OX*, as shown by the arrows in the figure.

But twice the area *OAR* is the moment of the resultant *AR*, and twice the areas *OAB* and *OAC* are the moments of the components *AB* and *AC* about *O*. Hence the moment of the resultant is equal to the algebraic sum of the moments of the components.

If we have a third displacement, velocity or acceleration at *A*, the resultant of this and *AR* would be the resultant for all three. Hence the principle holds for any number of components.

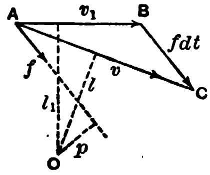


Again, let *AB*, *BC*, *CD* represent the components of a point *A*. Then the resultant is *AD*. Let *OY* be an axis and *XZ* a plane perpendicular to the axis. Let *ab*, *bc*, *cd* be the projections on this plane of the component velocities.

We have just proved that the moment of *ad* about *O* is the algebraic sum of the moments of the components *ab*, *bc*, *cd*.

But the moment of each of these about *O* is the moment of *AB*, *BC*, *CD* about the axis (page 60). Hence the moment of the resultant *AD* about the axis is equal to the algebraic sum of the moments of the components *AB*, *BC*, *CD*.

The moment of acceleration of a moving point relative to any fixed point in the plane of its motion is equal to the time-rate of change of the moment of its velocity about the same point.



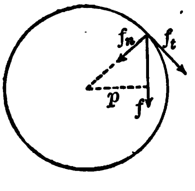
Let $AB = v_1$ be the instantaneous velocity of a point *A* and *f* its instantaneous acceleration. Then in any indefinitely small time *dt* the change of velocity is $BC = fdt$, and the resultant velocity is $AC = v$ in the plane of v_1 and *f*. Take a point *O* in the same plane and drop the perpendiculars l_1, l and p upon the directions of v_1, v and f.

Then, since the moment of the resultant is equal to the algebraic sum of the moments of the components, we have

$$vl = v_1l_1 + fdt \,.\, p, \text{ or } fp = \frac{vl - v_1l_1}{dt}.$$

If the path is a circle of radius *r*, then $l = l_1 = r$, and we have relative to the centre

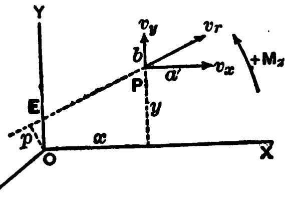
$$fp = f_tr.$$



We obtain the same result as follows:

Resolve the acceleration *f* into components f_t tangent to the circle and f_n normal. The latter component passes through the centre, and its moment is zero. We have then the moment of *f* equal to the moment of the other component, or

$$fp = f_tr.$$



General Analytical Determination of Resultant Velocity and Moment for Any Number of Concurring Component Velocities in the Same Plane.—Let the point *P* be given by the co-ordinates *x*, *y*. Let the component velocities of *P* be v_1, v_2, v_3, etc., all in the same plane XY and making with the axes of X and Y the angles α_1, β_1; α_2, β_2; α_3, β_3, etc.

Let v be the resultant velocity making the angles a and b with the axes of X and Y respectively.

For the component velocities parallel to X and Y we have

$$\left.\begin{aligned} v_x &= v_1\cos\alpha_1 + v_2\cos\alpha_2 + v_3\cos\alpha_3 + \ldots = \Sigma v\cos\alpha; \\ v_y &= v_1\cos\beta_1 + v_2\cos\beta_2 + v_3\cos\beta_3 + \ldots = \Sigma v\cos\beta. \end{aligned}\right\} . \qquad (1)$$

In these summations components towards the right or upwards are positive, towards the left or downwards negative.

The resultant velocity is then

$$v_r = \sqrt{v_x^2 + v_y^2}, \quad \ldots\ldots\ldots \quad (2)$$

making with the axes of X and Y angles a and b given by

$$\cos a = \frac{v_x}{v_r}, \quad \cos b = \frac{v_y}{v_r}; \quad \ldots\ldots \quad (3)$$

The moment of the resultant velocity with reference to O is the algebraic sum of the moments of the component velocities v_x and v_y. If p is the lever arm of v_r, we have, paying regard to the sign for direction of rotation, for the moment M_z about the axis OZ, or the moment in the plane XY,

$$M_z = v_r p = v_y x - v_x y \quad \ldots\ldots \quad (4)$$

Hence the lever-arm is

$$p = \frac{M_z}{v_r}. \quad \ldots\ldots\ldots \quad (5)$$

In these equations v_x, v_y, x and y are positive in the directions OX, OY and negative in the opposite directions. With these conventions the equations are general.

If M_z comes out positive, the direction of rotation about O is counter-clockwise; if negative, the direction of rotation is clockwise as shown in the figure.

In the first case the *line representative* is positive and therefore passes through O in the direction OZ. In the second case it has the contrary direction. In both cases, if we look along the line representative *towards the origin*, the direction of rotation will be seen as counter-clockwise.

Since v_r may be considered, so far as the moment at O is concerned, as acting at any point in its line of direction (page 60), let us take it acting at E, the intersection of the line of direction of v with the axis of Y. Then we have for the distance OE, $v_x \times OE = -M_z$, or $OE = -\frac{M_z}{v_x}$. The tangent of the angle which v_r makes with the axis of X is $\frac{v_y}{v_x}$. Hence the equation of the line of direction of the resultant velocity v is

$$y = \frac{v_y}{v_x}x - \frac{M_z}{v_x}. \quad \ldots\ldots\ldots \quad (6)$$

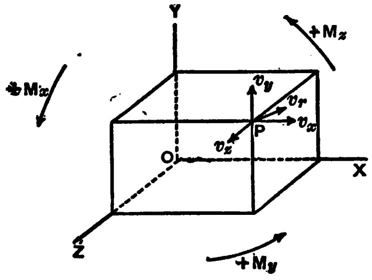
If in this equation we make $x = 0$, we find the ordinate of the point in which the direction of the resultant velocity v_r intersects the axis of Y, viz.,

$$OE = y' = -\frac{M_z}{v_x}. \quad \ldots\ldots\ldots \quad (7)$$

If we make $y = 0$, we find the abscissa of the point in which the direction of the resultant velocity v_r intersects the axis of X, viz.,

$$x' = \frac{M_z}{v_y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

General Analytic Determination of Resultant Velocity and Moment for Concurring Component Velocities not in the Same Plane.—



Let the point P be given by the space co-ordinates x, y, z, and let the component velocities of P be v_1, v_2, v_3, etc., making with the axes of X, Y, and Z the angles α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3, etc.

Let v_r be the resultant velocity making the angles a, b, c with the axes.

We have then for the component velocities parallel to X, Y, and Z

$$\left.\begin{aligned} v_x &= v_1 \cos \alpha_1 + v_2 \cos \alpha_2 + v_3 \cos \alpha_3 = \Sigma v \cos \alpha; \\ v_y &= v_1 \cos \beta_1 + v_2 \cos \beta_2 + v_3 \cos \beta_3 = \Sigma v \cos \beta; \\ v_z &= v_1 \cos \gamma_1 + v_2 \cos \gamma_2 + v_3 \cos \gamma_3 = \Sigma v \cos \gamma. \end{aligned}\right\} \quad . \quad . \quad (1)$$

In these summations components in the directions OX, OY, OZ are positive, in the opposite directions negative.

The resultant velocity is then

$$v_r = \sqrt{v_x^2 + v_y^2 + v_z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

making with the axes of X, Y and Z angles a, b and c given by

$$\cos a = \frac{v_x}{v_r}, \quad \cos b = \frac{v_y}{v_r}, \quad \cos c = \frac{v_z}{v_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The moment of the resultant velocity v_r with reference to O is the algebraic sum of the moments of the component velocities v_x, v_y and v_z.

We take the positive direction of rotation in each of the coordinate planes in the direction indicated by the arrows in the figure. Thus,

$$\left.\begin{array}{l} \text{rotation about } Z \text{ from } X \text{ to } Y \\ \text{“} \quad \text{“} \quad X \quad \text{“} \quad Y \text{ “ } Z \\ \text{“} \quad \text{“} \quad Y \quad \text{“} \quad Z \text{ “ } X \end{array}\right\} \text{are positive;}$$

in the contrary directions, negative.

We have then for the moments about the axes

$$\left.\begin{array}{l} \text{moment about } Z \text{ parallel to plane } XY, \; M_z = v_y x - v_x y; \\ \text{“} \quad \text{“} \quad X \quad \text{“} \quad \text{“} \quad \text{“} \quad YZ, \; M_x = v_z y - v_y z; \\ \text{“} \quad \text{“} \quad Y \quad \text{“} \quad \text{“} \quad \text{“} \quad ZX, \; M_y = v_x z - v_y x. \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

In these equations v_x, v_y, v_z and x, y, z are positive in the directions OX, OY, OZ, and negative in the opposite directions. With these conventions the equations are general; and if M_z M_x, M_y come out positive, we have rotation in each plane counter-clockwise as indicated by the figure; if negative, clockwise.

In the first case the line representatives pass through O and have the directions OZ, OX, OY. In the second case they have opposite directions through O. In any case the direction of rotation is always counter-clockwise *when we look along the line representative towards O.*

The equations of the projections of the line of direction of the resultant velocity v_r upon the co-ordinate planes are found as in the preceding Article, since each may be considered as acting at any point in its line of direction:

$$\left.\begin{array}{lll} \text{on plane } XY, & y = \dfrac{v_y}{v_x}x - \dfrac{M_z}{v_x}; \\ \text{“} \quad \text{“} \quad YZ, & z = \dfrac{v_z}{v_y}y - \dfrac{M_x}{v_y}; \\ \text{“} \quad \text{“} \quad ZX, & x = \dfrac{v_x}{v_z}z - \dfrac{M_y}{v_z}. \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (5)$$

If in these equations we make $z = 0$, we find the co-ordinates of the point in which the direction of the resultant velocity v_r pierces the plane XY, viz.,

$$x' = -\frac{M_y}{v_z}, \quad y' = \frac{M_x}{v_z}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

If we make $x = 0$, we have for the co-ordinates of the point where it pierces the plane YZ,

$$z' = \frac{M_y}{v_x}, \quad y' = -\frac{M_z}{v_x}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

If we make $y = 0$, we have for the co-ordinates of the point where it pierces the plane ZX,

$$z' = -\frac{M_x}{v_y}, \quad x' = \frac{M_z}{v_y}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

Combining the line representatives of the moments given by (4) we have

$$M_r = v_r p = \sqrt{M_x^2 + M_y^2 + M_z^2} \quad . \; . \; . \; . \; . \quad (9)$$

Hence

$$p = \frac{M_r}{v_r}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

The line representative for the resultant moment M_r passes through O and makes the angles d, e, f with the axes of X, Y, Z given by

$$\cos d = \frac{M_x}{M_r}, \quad \cos e = \frac{M_y}{M_r}, \quad \cos f = \frac{M_z}{M_r}. \quad . \; . \; . \; . \quad (11)$$

Looking along this line representative *towards O*, the direction of rotation is *always counter-clockwise.*

The projections of this line representative upon the co-ordinate planes make angles with the axes given as follows:

$$\left.\begin{array}{lr} \text{projection on } XY \text{ tangent of angle with } & X = \dfrac{M_y}{M_x}; \\ \text{“} \quad \text{“} \; YZ \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} & Y = \dfrac{M_z}{M_y}; \\ \text{“} \quad \text{“} \; ZX \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} & Z = \dfrac{M_x}{M_z}. \end{array}\right\} \; . \; . \quad (12)$$

If we make $v_z = 0$, we obtain the equations of the preceding article. If we make x, y, z zero, we have the equations of page 37 if we put v in place of d.

General Analytic Determination of Resultant Acceleration and Moment for Concurring Component Accelerations.—The equations of the last two Articles hold good for a point having component accelerations as well as for component velocities. We have only to substitute f in place of v.

General Analytic Determination of Resultant Displacement and Moment for Concurring Component Displacements.—The same equations hold for a point having component displacements. We have only to substitute d in place of v. If we then make $x, y, z = 0$ we have the equations of page 37.

EXAMPLES.

(1) *A point P given by the co-ordinates* $x = +3$ *ft.*, $y = +4$ *ft.*, $z = 0$ *has the component velocities* $v_1 = 40$, $v_2 = 50$, $v_3 = 60$ *ft. per sec., making the angles with X, Y and Z,* $\alpha_1 = 60°$, $\beta_1 = 150°$, $\gamma_1 = 90°$; $\alpha_2 = 120°$, $\beta_2 = 30°$, $\gamma_2 = 90°$; $\alpha_3 = 120°$, $\beta_3 = 150°$, $\gamma_3 = 90°$. *Find the resultant velocity and the resultant moment about the origin.*

Ans. The component velocities are in one plane and

$$v_x = +20 - 25 - 30 = -35 \text{ ft. per sec.};$$
$$v_y = -34.64 + 43.3 - 51.96 = -43.3 \text{ ft. per sec.}$$

The resultant $v_r = \sqrt{v_x^2 + v_y^2} = 55.67$ ft. per sec., making with the horizontal the angle $\cos a = \frac{v_x}{v_r} = \frac{-35}{55.67}$ or $a = 128° \, 57' \, 17''$, and with the vertical angle $\cos b = \frac{v_y}{v_r} = \frac{-43.3}{55.67}$, or $b = 141° \, 2' \, 43''$.

The moment of the resultant velocity v_r with reference to O is $M_z = -130 + 140 = +10$ sq. ft. per sec.

The direction of motion of the radius vector in the plane XY is therefore counter-clockwise.

The lever-arm $p = \frac{10}{55.67} =$ about 0.18 ft.

The equation of the line of direction of the resultant velocity is $y = 1.237x + 0.286$ ft.

The intercepts on the axes are $y' = +0.286$ ft., $x' = -0.231$ ft.

(2) *A point P given by co-ordinates* $x = +3$ *ft.*, $y = +4$ *ft.*, $z = +5$ *ft. has the component velocities* $v_1 = 40$, $v_2 = 50$, $v_3 = 60$ *ft. per sec., making the angles with X, Y, Z,* $\alpha_1 = 60°$, $\beta_1 = 100°$, γ_1 *obtuse*; $\alpha_2 = 100°$, $\beta_2 = 60°$, γ_2 *acute*; $\alpha_3 = 120°$, $\beta_3 = 100°$, γ_3 *acute. Find the resultant velocity and the resultant moment about the origin.*

Ans. We find the angles γ (page 12) by the equation

$$\cos^2 \gamma = -\cos(\alpha + \beta) \cos(\alpha - \beta).$$

Hence $\gamma_1 = 148° \, 2' \, 31''.7$, $\quad \gamma_2 = 31° \, 57' \, 28''.3$, $\quad \gamma_3 = 31° \, 57' \, 28''.3$.

$$v_x = +20 - 8.6824 - 30 = -18.6824 \text{ ft. per sec.}$$
$$v_y = -6.946 + 25 - 10.419 = +7.635 \text{ ft. per sec.}$$
$$v_z = -33.937 + 42.421 + 50.907 = +59.391 \text{ ft. per sec.}$$

The resultant velocity is

$$d_r = \sqrt{v_x^2 + v_y^2 + v_z^2} = 62.73 \text{ ft. per sec.,}$$

making with the axes of X, Y, Z angles given by

$$\cos a = \frac{-18.6824}{62.73}, \quad \text{or } a = 107^\circ\ 19'\ 36'';$$

$$\cos b = \frac{+7.635}{62.73}, \quad \text{or } b = \ 83^\circ\ \ 0'\ 33'';$$

$$\cos c = \frac{+59.391}{62.73}, \quad \text{or } c = \ 18^\circ\ 46'\ 42''.$$

The moments in the co-ordinate planes are

$$M_z = +22.905 + 74.7296 = +\ \ 97.6346 \text{ sq. ft. per sec.}$$

$$M_x = +237.564 - 38.175 = +199.389 \quad \text{“ “ “ “}$$

$$M_y = -93.412 - 178.173 = -271.585 \quad \text{“ “ “ “}$$

The resultant moment is

$$M_r = \sqrt{M_x^2 + M_y^2 + M_z^2} = 350.78 \text{ sq. ft. per sec.}$$

The line representative makes with the axes of X, Y, Z angles given by

$$\cos d = \frac{M_x}{M_r} = \frac{+199.389}{407.6}, \text{ or } d = \ 55^\circ\ 21'\ 37'';$$

$$\cos e = \frac{M_y}{M_r} = \frac{-271.585}{407.6}, \text{ or } e = 140^\circ\ 44'\ \ 8'';$$

$$\cos f = \frac{M_z}{M_r} = \frac{+97.6346}{407.6}, \text{ or } f = \ 78^\circ\ 50'\ 21''.$$

Looking along this line towards O, the motion of the radius vector is counter-clockwise.

The equations of the projections of the direction of the resultant velocity v_r upon the co-ordinate planes are:

on plane XY, $y = -0.408x + \ 5.226$ ft.;

on plane YZ, $z = +7.778y - 26\ 115$ ft.;

on plane ZX, $x = -0.314z + \ 4.572$ ft.

The point in which the direction of the resultant velocity pierces the plane XY is given by $x' = +4.572$ ft., $y' = +3.357$ ft.

The point in which the direction of the resultant velocity pierces the plane YZ is given by $y' = +5.226$ ft., $z' = +14.56$ ft.

The point in which the direction of the resultant velocity pierces the plane ZX is given by $z' = -26.115$ ft., $x' = +12.788$ ft.

(3) *A point given by the co-ordinates $x = +3$ ft., $y = +4$ ft., $z = 0$, has the component accelerations $f_1 = 40$, $f_2 = 50$, $f_3 = 60$ ft.-per-sec. per sec., making the angles with X, Y and Z, $\alpha_1 = 60^\circ$, β_1 obtuse, $\gamma_1 = 90^\circ$; $\beta_2 = 30^\circ$, α_2 obtuse, $\gamma_2 = 90^\circ$; $\alpha_3 = 120^\circ$, β_3 obtuse, $\gamma_3 = 90^\circ$. Find the resultant acceleration and the resultant moment about the origin.*

Ans. (page 64). The component accelerations are in one plane and

$$f_x = +20 - 25 - 30 = -35 \text{ ft.-per-sec. per sec.}$$

$$f_y = -34.64 + 43.3 - 51.96 = -43.3 \text{ ft.-per sec. per sec.}$$

The resultant $f_r = \sqrt{f_x^2 + f_y^2} = 55.67$ ft.-per-sec. per sec., making with the horizontal the angle $\cos a = \frac{f_x}{f} = \frac{-35}{55.67}$, or $a = 128° \, 57' \, 17''$, and with the vertical the angle $\cos b = \frac{f_y}{f} = \frac{-43.3}{55.67}$, or $b = 141° \, 2' \, 43''$.

The moment of the resultant acceleration with reference to O is $M_z = -130 + 140 = +10$ square feet-per-sec. per sec. The direction of motion of the radius vector in the plane XY is therefore counter-clockwise.

The lever-arm p of the resultant is $p = \frac{10}{55.67}$ = about 0.18 ft.

The equation of the line of direction of the resultant acceleration is $y = 1.237x + 0.286$ ft.

The intercepts on the axes are $y' = +0.286$ ft., $x' = -0.231$ ft.

(4) *A point given by the co-ordinates $x = +3$ ft., $y = +4$ ft., $= +5$ ft. has the component accelerations $f_1 = 40$, $f_2 = 50$, $f_3 = 60$ ft.-per-sec. per sec., making the angles with X, Y, Z, $\alpha_1 = 60°$, $\beta_1 = 100°$, γ_1 obtuse; $\alpha_2 = 100°$, $\beta_2 = 60°$, γ_2 acute; $\alpha_3 = 120°$, $\beta_3 = 100°$, γ_3 acute. Find the resultant acceleration and the resultant moment about the origin.*

Ans. (page 65). We find the angles γ (page 12) by

$$\cos^2\gamma = -\cos(\alpha + \beta)\cos(\alpha - \beta).$$

Hence $\gamma_1 = 148° \, 2' \, 31''.7$, $\gamma_2 = 31° \, 57' \, 28''.3$, $\gamma_3 = 31° \, 57' \, 28''.3$.

$f_x = +20 - 8.6824 - 30 = -18.6824$ ft.-per-sec. per sec;

$f_y = -6.946 + 25 - 10.419 = +7.635$ " " " " " ;

$f_z = -33.937 + 42.421 + 50.907 = +59.391$ " " " " " .

The resultant acceleration is

$$f_r = \sqrt{f_x^2 + f_y^2 + f_z^2} = 62.73 \text{ ft.-per-sec. per sec.},$$

making with the axes of X, Y and Z angles given by

$$\cos a = \frac{-18.6824}{62.73}, \text{ or } a = 107° \, 19' \, 36'';$$

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The moments in the co-ordinate planes are

$M_z = +22.905 + 74.7296 = +97.6346$ sq. ft.-per-sec. per sec.

$M_x = +237.564 - 38.175 = +199.389$ " " " " " "

$M_y = -93.412 - 178.173 = -271.585$ " " " " " "

The resultant moment is

$$M_r = \sqrt{M_x^2 + M_y^2 + M_z^2} = 350.78 \text{ sq. ft.-per-sec. per sec.}$$

The line representative makes with the axes of X, Y, Z angles given by

$$\cos d = \frac{M_x}{M} = \frac{+199.389}{407.6}, \text{ or } d = 55° \, 21' \, 37'';$$

$$\cos e = \frac{M_y}{M} = \frac{-271.585}{407.6}, \text{ or } e = 140° \, 44' \, 8'';$$

$$\cos f = \frac{M_z}{M} = \frac{+97.6346}{407.6}, \text{ or } f = 73° \, 50' \, 21''.$$

Looking along this line towards O, the motion of the radius vector is counter-clockwise.

The equations of the projection of the direction of the resultant acceleration f upon the co-ordinate planes are:

$$\begin{aligned} &\text{on plane } XY, \quad y = -0.408x + 5.226 \text{ ft.};\\ &\text{on plane } YZ, \quad z = +7.778y - 26.115 \text{ ft.};\\ &\text{on plane } ZX, \quad x = -0.314z + 4.572 \text{ ft.}\end{aligned}$$

The point in which the direction of the resultant acceleration pierces the plane XY is given by $x' = +4.572$ ft., $y' = +3.357$ ft.

The point in which the direction of the resultant acceleration pierces the plane YZ is given by $y' = +5.226$ ft., $z' = -14.56$ ft.

The point in which the direction of the resultant acceleration pierces the plane ZX is given by $z' = -26.115$ ft., $x' = +12.788$ ft.

(5) *A point given by the co-ordinates* $x = +3$ *ft.*, $y = +4$ *ft.*, $z = 0$ *has the component displacements* $d_1 = 40$ *ft.*, $d_2 = 50$ *ft.*, $d_3 = 60$ *ft.*, *making the angles with* X, Y *and* Z, $\alpha_1 = 60°$, β_1 *obtuse*, $\gamma_1 = 90°$; $\beta_2 = 30°$, α_2 *obtuse*, $\gamma_2 = 90$; $\alpha_3 = 120°$, β_3 *obtuse*, $\gamma_3 = 90°$. *Find the resultant displacement and moment about the origin.*

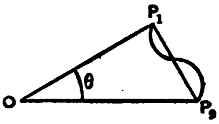
(6) *A point given by the co-ordinates* $x = +3$ *ft.*, $y = +4$ *ft.*, $z = +5$ *ft. has the component displacements* $d_1 = 40$ *ft.*, $d_2 = 50$ *ft.*, $d_3 = 60$ *ft.*, *making the angles with* X, Y *and* Z, $\alpha_1 = 60°$, $\beta_1 = 100°$, γ_1 *obtuse*; $\alpha_2 = 100°$, $\beta_2 = 60°$, γ_2 *acute*; $\alpha_3 = 120°$, $\beta_3 = 100°$, γ_3 *acute*. *Find the resultant displacement and moment about the origin.*

CHAPTER VII.

ANGULAR REVOLUTION OF A POINT. ANGULAR SPEED.

RATE OF CHANGE OF ANGULAR SPEED. EQUATIONS OF MOTION OF A POINT UNDER DIFFERENT RATES OF CHANGE OF ANGULAR SPEED. ANGULAR SPEED IN TERMS OF LINEAR VELOCITY. RATE OF CHANGE OF ANGULAR SPEED IN TERMS OF LINEAR. MOMENT OF LINEAR VELOCITY IN TERMS OF ANGULAR SPEED. MOMENT OF TANGENTIAL ACCELERATION IN TERMS OF RATE OF CHANGE OF ANGULAR SPEED. NORMAL ACCELERATION IN TERMS OF ANGULAR SPEED. MOTION IN A CIRCLE.

Angular Revolution of a Point about a Given Point.—When a point moves in any path whatever from the initial position P_1 to the final position P_2 in any given time, we have called the distance P_1P_2 the linear displacement (page 34).



If we choose any point in space O as a pole and draw the radius vector OP_1 to the initial and OP_2 to the final position, we call the angle $P_1OP_2 = \theta$ the **angular revolution** of the point P about O.

Since the angle θ is measured in radians, it is independent of the length of the radius vector, or the distance of P_1 and P_2 from O (page 5).

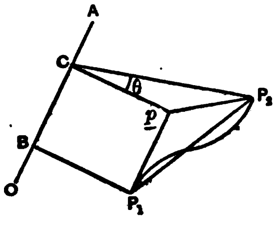
It is also independent of the position of the plane of revolution P_1OP_2 in space, or of the direction in space of the angular revolution.

It has, however, magnitude and sign (+) or (—), according as the radius vector moves in this plane in one direction or the other.

Angular revolution has then magnitude and sign, but not direction. It is therefore a scalar quantity like distance described by a point, and cannot be represented by a straight line.

The student must not confound angular revolution with "angular displacement," which, as we shall see hereafter (page 170), has like linear displacement, direction as well as sign and magnitude, and is therefore a vector quantity which can be represented by a straight line.

Angular Revolution of a Point about a Given Axis.—The angular revolution in any given time of a moving point about a given line or *axis* is the angle between perpendiculars from the initial and final positions of the point to the axis.



Thus let OA be a given axis, P_1 and P_2 the initial and final positions of the moving point, and P_1B, P_2C perpendiculars to OA. Then the angle between P_1B and P_2C is the angular revolution about OA, whatever the path between P_1 and P_2.

This angle is the same as the angle pCP_2, if we complete the rectangle CP_1. As the straight line pP_2 is thus the projection of the line P_1P_2 on the plane P_2Cp, we see that the angular revolution about the axis is the angular revolution of the projection p of P_1 about the point C.

Mean Angular Speed of a Point about a Given Point or Axis.—The angular revolution per unit of time is the **mean angular speed** of a point about a given point or axis.

Like angular revolution it has then magnitude and sign according to direction of motion in the plane of revolution, but is independent of the position in space of that plane. It is therefore a scalar quantity like linear speed (page 16).

When the mean angular speed varies with the time it is **variable**. When it has the same magnitude no matter what the interval of time it is **uniform**. A point moving with uniform angular speed evidently describes equal angles in equal times.

Instantaneous Angular Speed of a Point about a Given Point or Axis.—The limiting value of the mean angular speed when the interval of time is indefinitely small is the **instantaneous angular speed**. If the instantaneous angular speed is variable, the mean angular speed has different values for equal intervals of time.

The term angular speed always signifies instantaneous angular speed unless otherwise specified.

Angular speed like mean angular speed is therefore a scalar quantity, having magnitude and sign according to the direction of motion in the plane of revolution, but independent of the position of this plane in space.

The student must not confound angular speed with "angular velocity," which, as we shall see hereafter (page 174), has direction as well as sign and magnitude and is therefore a vector quantity.

Numeric Equations of Angular Speed.—The unit of angular speed is evidently one *radian per second*. We denote the magnitude of the angular speed thus measured by the letter ω.

If then θ_1 is the angle measured in the plane of revolution from any fixed line to the initial position of the radius vector and θ to the final position of the radius vector, we have for the mean angular speed

$$\omega = \frac{\theta - \theta_1}{t} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

When the interval of time is indefinitely small, we have in the Calculus notation, for the instantaneous angular speed,

$$\omega = \frac{d\theta}{dt} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2)$$

Sign of Angular Speed.—These equations are precisely the same as equations (1) and (2), page 16, simply substituting θ for s. The sign follows the same rule. Thus when the angle is increasing the value of ω is positive (+), and when decreasing it is negative (—).

Equation (1) is thus general if we take angles in any one direction in the plane of revolution measured from a fixed line in that plane as positive, and in the opposite direction as negative.

Angular speed, then, whether uniform or variable, mean or instantaneous, is independent of direction in space. It is entirely comparable to linear speed (page 15).

Change of Angular Speed.—When the angular speed of a point varies, the difference between the final and initial instantaneous speeds for any interval of time is the **integral** change of angular speed.

Mean Rate of Change of Angular Speed.—The integral change of angular speed per unit of time is the **mean rate of change** of angular speed.

When the mean rate of change varies with the time it is **variable**. When it has the same magnitude no matter what the interval of time it is uniform.

Instantaneous Rate of Change of Angular Speed.—The limiting value of the mean rate of change of angular speed when the interval of time is indefinitely small is the **instantaneous** rate of change of angular speed.

Rate of change of angular speed should always be understood as meaning instantaneous rate of change unless otherwise specified.

Rate of change of angular speed may be zero, uniform or variable. When it is zero the angular speed is uniform and the same as the mean speed for any interval of time.

When it is uniform the rate of change of angular speed is the same as the mean rate of change for any interval of time.

When it is variable the mean rate of change has different values for equal intervals of time.

Numeric Equations of Rate of Change of Angular Speed.—The unit of rate of change of angular speed is then one radian-per-sec. per sec. We denote its magnitude thus measured by the letter α.

If then ω_1 is the initial and ω the final instantaneous angular speed, we have for the mean rate of change of angular speed

$$\alpha = \frac{\omega - \omega_1}{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for the instantaneous rate of change of angular speed

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Sign of Rate of Change of Angular Speed.—These equations are precisely the same as equations (1) and (2), page 25, simply substituting ω for v and θ for s and α for a. The value then of α is positive (+) when the angular speed increases and negative (—) when it decreases during the time.

It is evident that this sign has no reference to position in space. Rate of change of angular speed is then a scalar quantity.

The student must not confound rate of change of angular speed with "angular acceleration," which, as we shall see hereafter (page 175), has direction as well as sign and magnitude and is therefore a vector quantity.

Equations of Motion of a Point under Different Rates of Change of Angular Speed.—We have equations precisely similar to those for linear speed, page 27. We have only to substitute ω for v, α for a, θ for s.

(*a*) **Rate of Change of Angular Speed Zero.**—In this case if θ_1 is the initial angle measured in the plane of revolution of the radius vector from a fixed line in that plane, and θ the final angle, we have

$$\omega = \frac{\theta - \theta_1}{t}, \quad \text{or} \quad \omega t = \theta - \theta_1. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Revolution in any one direction in the plane being taken as positive, in the other direction it is negative. Then if ω comes out (+) it denotes revolution in the assumed positive direction; if (—), in the opposite direction. If t comes out (+) it denotes time after, if (—) time before, the start.

(*b*) **Rate of Change of Angular Speed Uniform.**—When the rate of change of angular speed is uniform, the instantaneous rate of change of angular speed at any instant is equal to the mean rate of change for any interval of time.

If ω_1 and ω are the initial and final instantaneous angular speeds, we have then for the rate of change of angular speed

$$\alpha = \frac{\omega - \omega_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of α is (+) when the angular speed increases and (−) when it decreases during the time.

From equation (2) we have

$$\omega = \omega_1 + \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The average angular speed is

$$\frac{\omega + \omega_1}{2} = \omega_1 + \frac{1}{2}\alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The angle described in the time t is

$$\theta - \theta_1 = \frac{\omega + \omega_1}{2}t = \omega_1 t + \frac{1}{2}\alpha t^2. \quad . \quad . \quad . \quad (5)$$

Inserting the value of t from (2) we have

$$\theta - \theta_1 = \frac{\omega^2 - \omega_1^2}{2\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence $$\omega^2 = \omega_1^2 + 2\alpha(\theta - \theta_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In applying these formulas, α is positive (+) when the angular speed increases and negative (−) when it decreases during the time, without regard to direction of revolution.

If angles θ, θ_1 in one direction are taken as (+), angles in the opposite direction are (−).

Angular speeds ω, ω_1 are positive (+) when motion is in the assumed positive direction, and negative (−) when in the other direction. A positive t denotes time after the beginning of motion, and a negative t time before.

[(*c*) **Rate of Change of Angular Speed Variable.**]—If the rate of change of angular speed is variable, we have from (1), in Calculus notation,

$$\omega = \frac{d\theta}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

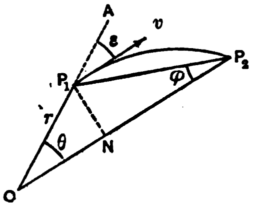
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and from (8),

$$\theta - \theta_1 = \int_{t=0}^{t=t} \omega dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The preceding equations can be directly deduced from these as on page 28.

Angular Speed in Terms of Linear Velocity.—If a point moves from P_1 to P_2 in any path in the time t, the linear displacement in that time is the chord P_1P_2.



If we take O as a pole, the angular displacement of revolution is $P_1OP_2 = \theta$. The mean linear velocity is

$$\frac{\text{chord } P_1P_2}{t},$$

and the mean angular speed is $\frac{\theta}{t}$.

From P_1 draw P_1N perpendicular to OP_2. Then if angle $P_1P_2N = \phi$, we have

$$P_1N = \text{chord } P_1P_2 \,.\, \sin\phi.$$

Dividing by t, we have

$$\frac{P_1N}{t} = \frac{\text{chord } P_1P_2 \,.\, \sin\phi}{t}.$$

But $P_1N = r\sin\theta$, where r is the radius vector OP_1. Hence

$$\frac{r\sin\theta}{t} = \frac{\text{chord } P_1P_2 \,.\, \sin\phi}{t}.$$

If now the time is indefinitely small, $\frac{\text{chord } P_1P_2}{t}$ becomes the instantaneous velocity v, and ϕ becomes the angle $AP_1v = \epsilon$ between the radius vector OP_1 and the instantaneous velocity v, and $\sin\theta$ becomes θ, and $\frac{\theta}{t}$ becomes the instantaneous angular speed ω. Hence

$$r\omega = v\sin\epsilon, \quad \text{or} \quad \omega = \frac{v\sin\epsilon}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

In general, then, whatever the path or wherever the pole,

The magnitude of the angular speed at any point is equal to the magnitude of the component of the linear velocity at that point perpendicular to the radius vector, divided by the magnitude of the radius vector.

If the pole O is taken at the centre of curvature, so that OP_1 is equal to the radius of curvature ρ, then $\epsilon = 90°$ and we have $\rho\omega = v$ or $\omega = \frac{v}{\rho}$.

Rate of Change of Angular Speed in Terms of Tangential Linear Acceleration.—If $f_t = a$ is the magnitude of the linear tangential acceleration or rate of change of speed at any point, then we can prove, precisely as in the preceding Article, that

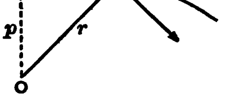
$$\alpha = \frac{f_t\sin\epsilon}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where α is the magnitude of the rate of change of angular speed.

Hence *the magnitude of the rate of change of angular speed at any point is equal to the magnitude of the component of the linear tangential acceleration at that point perpendicular to the radius vector, divided by the magnitude of the radius vector.*

If the pole O is taken at the centre of curvature, $\epsilon = 90°$ and we have $\rho\alpha = f_t$ or $\alpha = \frac{f_t}{\rho}$.

Moment of Linear Velocity in Terms of Angular Speed.—We can resolve the linear velocity v at the point P into a component $v\cos\epsilon$ along the radius vector and a component $v\sin\epsilon$ perpendicular to the radius vector. The moment of the first relative to the pole is zero. Since the moment of v is equal to the algebraic sum of the moments of its components (page 62), if we take moments about the pole, we have



$$vp = v\sin\epsilon \,.\, r.$$

But we have just seen that $v \sin \epsilon = r\omega$. Hence

$$vp = r^2\omega. \quad \ldots\ldots\ldots\ldots \quad (3)$$

That is, *the magnitude of the moment of the linear velocity at any point relative to the pole is equal to the magnitude of the angular speed at that point, multiplied by the square of the magnitude of the radius vector.*

Since $v \sin \epsilon$ is the normal component of v, $v \sin \epsilon \,.\, r$ is *twice the areal velocity of the radius vector* (page 61).

Moment of Linear Tangential Acceleration in Terms of Rate of Change of Angular Speed.—We can resolve the tangential acceleration f_t into components along and perpendicular to the radius vector and thus obtain, precisely as in the preceding Article,

$$fp = f_t p_t = r^2\alpha. \quad \ldots\ldots\ldots\ldots \quad (4)$$

Hence *the magnitude of the moment of the linear tangential acceleration at any point relative to the pole is equal to the magnitude of the rate of change of angular speed at that point, multiplied by the square of the magnitude of the radius vector.*

Since $f_t \sin \epsilon$ is the normal component of f_t, $f_t \sin \epsilon \,.\, r = r^2\alpha$ is twice the areal acceleration of the radius vector (page 61).

Normal Linear Acceleration in Terms of Angular Speed.—We have seen (page 53) that when a point moves in any path, the magnitude of the normal acceleration f_n is given by $f_n = \frac{v^2}{\rho}$, where ρ is the radius of curvature.

If we take the pole at the centre of curvature, then, we have $v = \rho\omega$, and hence

$$f_n = \frac{v^2}{\rho} = \rho\omega^2 = v\omega. \quad \ldots\ldots\ldots\ldots \quad (5)$$

The magnitude of the normal linear acceleration at any point is equal to the magnitude of the radius of curvature at that point, multiplied by the magnitude of the square of the angular speed, or to the velocity $v\omega$ in the hodograph (page 52).

Since for any path $r\omega = v \sin \epsilon$, we have

$$v = \frac{r\omega}{\sin \epsilon}.$$

Hence, in general,

$$f_n = v\omega = \rho\omega^2 = \frac{v^2}{\rho} = \frac{r^2\omega^2}{\rho \sin^2 \epsilon},$$

where r is any radius vector when the pole is not at the centre of curvature, and ϵ is the angle of v with this radius vector.

Motion in a Circle.—For a point moving in a circle we have $\epsilon = 0$ and $r = \rho$. Hence from (1), page 75, we have

$$v = r\omega, \quad \text{or} \quad \omega = \frac{v}{r}. \quad \ldots\ldots\ldots\ldots \quad (1)$$

If r is unity, we have the numeric equation $\omega = v$, that is, *the number of units of angular speed is equal to the number of units of linear speed at distance unity.*

From (2), page 139, we have

$$f_t = r\alpha, \quad \text{or} \quad \alpha = \frac{f_t}{r}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

If r is unity, we have the numeric equation $\alpha = f_t$, that is, *the number of units of rate of change of angular speed is equal to the number of units of linear tangential acceleration at distance unity.*

From (5), page 76, we have in any case

$$f_n = v\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or the *normal linear acceleration is equal to the velocity in the hodograph* (page 52). Inserting the value of v from (1),

$$f_n = r\omega^2 = \frac{v^2}{r}, \quad \text{or} \quad \omega^2 = \frac{f_n}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If r is unity, we have the numeric equation $f_n = v^2 = \omega^2$, that is, *the number of units of the normal linear acceleration is equal to the square of the number of units of linear velocity at distance unity; or the square of the number of units of angular speed is equal to the number of units of the normal linear acceleration at distance unity.*

We have also for the total resultant linear acceleration

$$f = \sqrt{f_t^2 + f_n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since the component f_n passes through the centre, the moment of f relative to the centre is equal to the moment of f_t. Hence

$$vr = r^2\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$fp = f_t r = r^2\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

give the moments of v and f with respect to the centre.

If the point starts from rest and acquires the velocity v in the time t, under constant tangential acceleration, we have $f_t = \frac{v}{t}$, $\alpha = \frac{\omega}{t}$.

Graphic Representation of Rate of Change of Angular Speed.—We can represent intervals of time by distances laid off horizontally and the corresponding angular speeds by distances laid off vertically and thus obtain the same diagrams as for linear speed given on page 29.

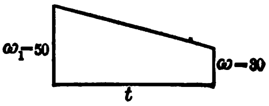
EXAMPLES.

(1) *The angular speed of a point moving in a plane about some assumed point changes from* 50 *to* 30 *radians per sec. in passing through* 80 *radians. Find the constant rate of change of angular speed and the time of motion.*

Ans. $\alpha = \frac{\omega^2 - \omega_1^2}{2(\theta - \theta_1)} = -10$ radians-per-sec. per sec. The minus sign denotes decreasing speed. $t = \frac{\omega - \omega_1}{\alpha} = 2$ sec.

(2) *Draw a figure representing the motion in the preceding example, and deduce the results directly from it.*

Ans. Average speed $= \frac{50 + 30}{2} = 40$ radians per sec. Hence $40t = 80$ or $t = 2$ seconds. Also $\alpha = \frac{30 - 50}{2} = -10$ radians-per-sec. per sec.



(3) *A point moving in a plane has an initial speed of* 60 *radians per sec. about an assumed point and a rate of change of speed of* + 40 *radians-per-sec. per sec. Find the speed after* 8 *sec.; the time required to describe* 300 *radians; the change of speed while describing that angle; the final speed.*

Ans. See Example (9), page 31.

(4) *If the motion in the last example is retarded, find* (*a*) *the angular revolution from the start to the turning-point;* (*b*) *the angle described from the start after* 10 *sec.; the speed acquired and the angle between the final and initial positions;* (*c*) *the angle described during the time in which the speed changes to* − 90 *radians per sec., and this time;* (*d*) *the time required by the moving point to return to the initial position.*

Ans. See Example 10, page 32.

(5) *A point moving in a plane describes about a fixed point angles of* 120 *radians,* 228 *radians and* 336 *radians in successive tenths of a second. Show that this is consistent with uniform rate of change of angular speed, and find this rate.*

Ans. $a = 10800$ radians-per-sec. per sec.

(6) *Two points A and B move in the circumference of a circle with uniform angular speeds* ω *and* ω'. *The angle between them at the start is* a. *Find the time of the nth meeting, the angles described by A and B, and the interval of time between two successive meetings.*

Ans. See Example (21), page 21.

Time of the nth meeting, $t_n = \dfrac{\pm a + (n-1)2\pi}{\omega \pm \omega'}$.

Angle described by A is $\theta = \omega t_n$.

" " " B is $\mp a$.

Interval of time between two successive conjunctions is

$$t_2 - t_1 = t_3 - t_2 = \frac{2\pi}{\omega \pm \omega'},$$

where we take the (+) or (−) sign for a according as B is in front of or behind A at start, and (+) or (−) sign for ω' according as the points move in opposite or the same directions.

(7) *What is the angular speed of a fly-wheel* 5 *ft. in diameter which makes* 30 *revolutions per minute, and what is the linear velocity of a point on its circumference? Also find its linear normal acceleration and the moment of its velocity with reference to the centre.*

Ans. π radians per sec.; 2.5π ft. per sec., tangent to circ.; $2.5\pi^2$ ft.-per-sec. per sec.; 6.25π sq. ft. per sec.

(8) *Find the linear and angular speed of a point on the earth's equator, taking radius* 4000 *miles; also the linear normal acceleration.*

Ans. 1535.9 ft. per sec.; $\frac{\pi}{12}$ radians, or 15° per hour; 0.112 ft.-per-sec. per sec.

(9) *The angular speed of a wheel is* $\frac{3}{4}\pi$ *radians per sec. Find the linear speed of points at a distance of* 2 *ft.,* 4 *ft. and* 10 *ft. from the centre, also the linear normal acceleration.*

Ans. $\frac{3}{2}\pi$, 3π, 7.5π ft. per sec.

$\frac{9}{8}\pi^2$, $\frac{9}{4}\pi^2$, $\frac{45}{8}\pi^2$ ft.-per-sec. per sec.

(10) *If the linear speed of a point at the equator is v, find the speed linear and angular at any latitude λ.*

Ans. $v \cos \lambda$; $\frac{\pi}{12}$ radians per hour, or 15° per hour.

(11) *A point moves with uniform velocity v. Find at any instant its angular speed about a fixed point whose distance from the path is a.*

Ans. $\frac{va}{r^2}$ radians per sec., where r is the radius vector. Uniform velocity means uniform speed in a straight line (page 43).

Hence the angular speed of a point moving with uniform speed in a straight line is inversely proportional to the square of the distance of the point from a fixed point not in the line.

(12) *The speed of the periphery of a mill-wheel* 12 *feet in diameter is* 6 *feet per sec. How many revolutions does the wheel make per sec.?*

Ans. $\frac{1}{2\pi}$ revolutions.

(13) *The time is between* 5 *and* 6 *o'clock, and the hour and minute hands are together. What is the time?*

Ans. 5 h. 27 m. 16 sec. (see Example (6)).

(14) *Express in degrees and radians the angle made by the hands of a clock at* 3.35 *o'clock.*

Ans. 102.5 deg., 1.79 radians.

(15) *Find the multiplier for changing revolutions per minute into radians per second.*

Ans. 0.10472 rad. per sec. = 1 rev. per min.

(16) *The minute and second hands point in the same direction at* 12 *o'clock. When do they next point in the same direction?*

Ans. 1 min. $1\frac{1}{59}$ sec. after twelve. (See Example (6)).

(17) *Two clocks are together at XII. When the first comes to I, it has lost a second; when the second comes to I, it has gained a second. How far are they apart in* 12 *hours?*

Ans. 24 secs.

(18) *Two men start together to walk around a circular course, one taking* 75 *minutes to the round, the other* 90. *When will they be together again at the starting-point?*

Ans. 7.5 hours. (See Example (6)).

(19) *The hour-hand of a watch is* $\frac{3}{7}$ *of an inch long, the minute-hand* $\frac{4}{5}$ *of an inch, and the second-hand* $\frac{1}{3}$ *of an inch. Compare the lineal speeds of their points and the angular speeds.*

Ans. 5 : 112 : 2800; 1 : 12 : 720.

(20) *Deduce the equivalent of longitude for one minute of time and for one second of time.*

Ans. 15′ to 1 min., 15″ to 1 sec.

(21) *The diameter of the earth is nearly* 8000 *miles. Required the circumference at the equator and the linear speed at latitude* 60°.

Ans. 25000 miles; 521 miles per hour.

(22) *The wheel of a bicycle is* 52 *inches in diameter and performs* 5040 *revolutions in a journey of* 65 *minutes. Find the speed in miles per hour; the angular speed of any point about the axle; the areal velocity of a spoke; the relative velocity of the highest point with respect to the centre.*

Ans. 12 miles per hour; 8 : 12 radians per sec.; 19.06 sq. ft. per sec.; 12 miles per hour.

(23) *In going* 120 *yards the front wheel of a carriage makes six revolutions more than the hind wheel. If each circumference were a yard longer, it would make only* 4 *revolutions more. Find the circumference of each wheel.*

Ans. 4 yards and 5 yards.

(24) *If the velocity of a point is resolved into several components in one plane, show that its angular speed about any fixed point in the plane is the sum of the angular speeds due to the several components.*

(25) *A point moves with uniform speed v in a circle of radius r. Show that its angular speed about any point in the circumference is $\frac{v}{2r}$.*

(26) *Show that the angular speed of the earth about the sun is proportional to the apparent area of the sun's disk.* [*The radius vector from the sun to the earth sweeps over equal areas in equal times.*]

(27) *A point P moves in a parabola with constant angular speed about the focus S. Show that its linear speed is proportional to $\overline{SP}^{\frac{3}{2}}$.*

(28) *A point starting from rest moves in a circle with a constant rate of change of angular speed of* 2 *radians-per-sec. per sec. Find the angular speed at the end of* 20 *sec. and the angular displacement of revolution; also the linear speed and distance described and the number of revolutions; also the linear tangential acceleration and the normal linear acceleration at the end of* 20 *sec.*

Ans. 40 rad. per sec.; 400 radians; $40r$ ft. per sec.; $400r$ ft.; $\frac{400}{2\pi}$ revolutions; $2r$ ft.-per-sec. per sec. tangential acceleration; $1600r$ ft.-per-sec. per sec. normal acceleration.

(29) *A point moving with uniform rate of change of angular speed in a circle is found to revolve at the rate of* $8\frac{1}{4}$ *revolutions in the eighth second after starting and* $7\frac{1}{4}$ *revolutions in the thirteenth second after starting. Find its initial angular speed and its uniform rate of change of angular speed; also the initial linear speed and rate of change of speed; also the initial normal acceleration.*

Ans. 20.2π radians per sec.; -0.4π radians-per-sec. per sec.; $20.2\pi r$ ft. per sec.; $-0.4\pi r$ ft.-per-sec. per sec.; $408.04\pi^2 r$ ft.-per-sec. per sec.

(30) *A point starts from rest and moves in a circle with a uniform rate of change of angular speed of* 18 *radians-per-sec. per sec. Find the time in which it makes the first, second and third revolutions.*

Ans. $\frac{\sqrt{2\pi}}{3}$, $\frac{2\sqrt{\pi}-\sqrt{2\pi}}{3}$, $\frac{\sqrt{6\pi}-2\sqrt{\pi}}{3}$ secs.

CHAPTER VIII.

DIFFERENTIAL EQUATIONS OF MOTION OF A POINT.*

Free Motion of a Point—Rectangular Co-ordinates.—Let a moving point have a position at any instant given by the co-ordinates x, y and z, and let the distance described in the interval of time dt be ds, and let the direction of motion make the angles α, β, γ with X, Y, Z. Then we have

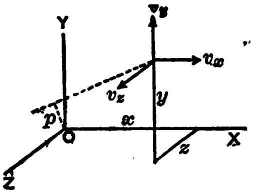
$$\cos\alpha = \frac{dx}{ds}, \quad \cos\beta = \frac{dy}{ds}, \quad \cos\gamma = \frac{dz}{ds}. \quad . \quad . \quad . \quad . \quad (1)$$

The magnitude of the velocity is

$$v = \frac{ds}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and its components in the direction of the axes are

$$\left.\begin{aligned} v_x &= v\cos\alpha = \frac{ds}{dt}\cos\alpha = \frac{dx}{dt}; \\ v_y &= v\cos\beta = \frac{ds}{dt}\cos\beta = \frac{dy}{dt}; \\ v_z &= v\cos\gamma = \frac{ds}{dt}\cos\gamma = \frac{dz}{dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$



We have v_x positive towards the right, negative towards the left; v_y positive upwards and negative downwards; v_z positive in the direction OZ, negative in the opposite direction (page 44).

Squaring equations (3) and adding, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have

$$v^2 = v_x^2 + v_y^2 + v_z^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, \quad . \quad . \quad (4)$$

or

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let the acceleration be f and its components in the direction of the axes X, Y, Z be f_x, f_y, f_z, then we have

$$f_x = \frac{d^2x}{dt^2}, \quad f_y = \frac{d^2y}{dt^2}, \quad f_z = \frac{d^2z}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

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* This Chapter must be omitted by those not familiar with the Calculus.

The acceleration f is then

$$f = \sqrt{f_x^2 + f_y^2 + f_z^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2} \quad . \quad . \quad . \quad (7)$$

We have f_x positive towards the right, negative towards the left; f_y positive upwards, negative downwards; f_z positive in the direction OZ, negative in the opposite direction.

The tangential acceleration $f_t = a$ is the rate of change of speed, or

$$f_t = a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Differentiating (4) and substituting (6), we have

$$\left.\begin{aligned} vdv &= f_x dx + f_y dy + f_z dz. \\ \text{Hence} \qquad & \\ v^2 &= 2\int (f_x dx + f_y dy + f_z dz) + \text{Const.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

Dividing by ds, since $v = \frac{ds}{dt}$, and $dv = \frac{d^2s}{dt}$, we have from (8)

$$\begin{aligned} f_t = a = \frac{dv}{dt} = \frac{d^2s}{dt^2} &= f_x \frac{dx}{ds} + f_y \frac{dy}{ds} + f_z \frac{dz}{ds} \\ &= \frac{dx}{ds} \cdot \frac{d^2x}{dt^2} + \frac{dy}{ds} \cdot \frac{d^2y}{dt^2} + \frac{dz}{ds} \cdot \frac{d^2z}{dt^2}. \quad . \quad (10) \end{aligned}$$

The normal acceleration f_n, as we have seen (page 76), is $f_n = \frac{v^2}{\rho}$, where ρ is the radius of curvature at the point. We have from analytical Geometry

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

Hence

$$f_n = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \rho\left(\frac{ds}{dt}\right)^2 \left\{ \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \right\}, \quad . \quad . \quad (11a)$$

and the acceleration f is

$$f = \sqrt{f_t^2 + f_n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11b)$$

If we denote by δ the angle which the acceleration f makes with the radius of curvature ρ, and by ϵ the angle which it makes with the tangent to the curve, we have $\delta + \epsilon = 90°$ and

$$\tan \epsilon = \frac{v^2}{\rho f_t} = \frac{\left(\frac{ds}{dt}\right)^2}{\rho\left(\frac{d^2s}{dt^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The moment of the velocity about the origin is the sum of the moments of the components. The moments in planes parallel to XY, YZ, ZX are:

$$\left.\begin{aligned} \text{about } z, \quad M_z &= \frac{dy}{dt} \cdot x - \frac{dx}{dt} \cdot y; \\ \text{“} \quad x, \quad M_x &= \frac{dz}{dt} \cdot y - \frac{dy}{dt} \cdot z; \\ \text{“} \quad y, \quad M_y &= \frac{dx}{dt} \cdot z - \frac{dz}{dt} \cdot x. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13a)$$

The moment about the origin of the resultant velocity v if p is its lever-arm is then

$$vp = M = \sqrt{M_x^2 + M_y^2 + M_z^2}. \quad . \quad . \quad . \quad . \quad . \quad (13b)$$

The line representative of this moment makes the angles d, e, f with the axes of X, Y, Z given by

$$\cos d = \frac{M_x}{M}, \quad \cos e = \frac{M_y}{M}, \quad \cos f = \frac{M_z}{M}. \quad . \quad . \quad . \quad . \quad (14)$$

Looking along this line representative towards the origin, the direction of rotation is always counter-clockwise.

In the same way the moment of the acceleration about the origin is the sum of the moments of the components. We have then precisely the same equations as (13), (14), only we put $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ in place of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ in order to find the moments in the co-ordinate planes.

Application of the Preceding Formulas.—Equations (3) and (6) are the general equations by which the motion of a point is determined. Applications of the use of the equations just deduced will be given hereafter. We can only indicate here the general application.

If $z = 0$, we have motion of a point in a plane only. The corresponding equations are at once obtained by making z and dz zero wherever they occur in the general equations.

If we also make $\beta = 0$ and $y = 0$, we have motion along the axis of x only. Hence taking $x = s$, we have from (3)

$$v = \frac{ds}{dt}; \quad \text{from (6)}, f = f_t = a = \frac{d^2s}{dt^2};$$

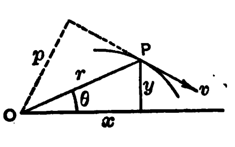
which are equations (8), (9) of page 51.

If the velocity v in any case is given, it can be resolved by (3) into its components v_x, v_y, v_z. Then by differentiating as indicated by (6) the components f_x, f_y, f_z of the acceleration f can be found, and the acceleration f can then be found by (7).

If the component accelerations are given, we find by integration the component velocities and then the resultant velocity.

If the path is required, each of equations (6) must be integrated twice, thus introducing two constants of integration for each. The constants of the first integration will depend on the initial velocity, those of the second on the initial position. We thus obtain equations involving x, y, z and t, and by eliminating t we obtain an equation between x and y, or y and z, or z and x, that is, the equation of the projection of the path on the co-ordinate planes.

Differential Polar Equations for Motion of a Point in a Plane.—Let x and y be the rectangular co-ordinates, and r and θ the polar co-ordinates, of a point P in a plane. Then



$$x = r\cos\theta, \quad y = r\sin\theta. \quad . \quad . \quad . \quad . \quad (15)$$

Differentiating and dividing by dt, we have

$$v_x = \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}; \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$v_y = \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Squaring and adding, since $\sin^2\theta + \cos^2\theta = 1$, we have for the magnitude of the velocity

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If r is constant, the path is a circle. In this case $\frac{dr}{dt}$ is zero and $v = r\frac{d\theta}{dt} = r\omega$, where ω is the angular speed (page 76).

The velocity along the radius vector is

$$\frac{dr}{dt} = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The velocity perpendicular to the radius vector is

$$r\frac{d\theta}{dt} = \frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Since by (6) $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ are the horizontal and vertical components of the acceleration, we have, by differentiating (16) and (17),

$$f_x = \frac{d^2x}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\cos\theta - \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\sin\theta; \quad (21)$$

$$f_y = \frac{d^2y}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\sin\theta + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\cos\theta. \quad (22)$$

The acceleration *along* the radius vector is then $f_x\cos\theta + f_y\sin\theta$, or

$$\frac{d^2y}{dt^2}\sin\theta + \frac{d^2x}{dt^2}\cos\theta = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If r is constant, the path is a circle. In this case $\frac{d^2r}{dt^2}$ is zero, and the acceleration along the radius vector is $f_n = -r\omega^2$, where ω is the angular speed (page 76). The $(-)$ sign denotes direction towards the centre (page 50).

The acceleration *perpendicular* to the radius vector is

$$f_y\cos\theta - f_x\sin\theta,$$

or

$$\frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If r is constant, the path is a circle, and $\frac{dr}{dt}$ is zero, and the acceleration perpendicular to the radius vector is $f_t = r\alpha$, where α is the rate of change of angular speed (page 76).

Equation (24) may be written

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = \frac{1}{r\,dt}d\left(r^2\frac{d\theta}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

From equation (18*a*) we have, by inserting the values of x, y and $\frac{dx}{dt}$,

$\frac{dy}{dt}$ from (16), (17), for the moment of the velocity with reference to the pole, if p is the lever-arm,

$$vp = M_s = r^2\frac{d\theta}{dt} = r^2\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where ω is the angular speed (page 76).

From equations (14) and (21) and (22) we have in like manner, for the moment of the acceleration,

$$fp = r\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

We see from (25) that this may also be written

$$fp = \frac{d\left(r^2\frac{d\theta}{dt}\right)}{dt} = f_t p_t = r^2\alpha, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where f_t is the tangential acceleration and α is the rate of change of angular speed (page 76).

Applications of the use of these formulas will be made hereafter.

General Polar Equations of Motion of a Point in a Plane—Acceleration Central.—When the acceleration is always directed to or from a fixed point it is called **central acceleration**, and the fixed point is called the **centre of acceleration**. Let this fixed point be the pole.

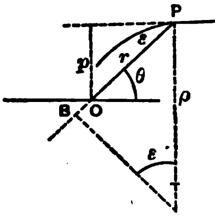
Then, since the direction of the acceleration always passes through the pole, its moment with reference to the pole is zero, and we have from (28)

$$d\left(r^2\frac{d\theta}{dt}\right) = 0, \quad \text{or} \quad r^2\frac{d\theta}{dt} = c, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where c is a constant of integration.

Now $\frac{d\theta}{dt} = \omega =$ angular speed, and from page 75 we have

$$r\omega = v\sin\epsilon,$$



where ϵ is the angle which the velocity at any point makes with the radius vector. Therefore

$$r^2\frac{d\theta}{dt} = r^2\omega = rv\sin\epsilon = c.$$

From page 76 we see that $r^2\omega$ is the moment of the velocity and is equal to twice the areal velocity of the radius vector.

Hence in central acceleration, *the moment of the velocity about the pole is constant, the area described by the radius vector in a unit of time is constant, and the radius vector therefore describes equal areas in equal times.*

The constant c is twice the area described by the radius vector in a unit of time (page 61).

If p is the perpendicular let fall from the pole upon the direction of the velocity, we have

$$vp = r^2\frac{d\theta}{dt} = r^2\omega = rv\sin\epsilon = c. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

From (30) we have

$$\frac{d\theta}{dt} = \omega = \frac{c}{r^2} \quad \text{and} \quad v = \frac{c}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Hence for central acceleration the angular speed at any point of the path is inversely as the square of the radius vector, and the linear velocity at that point is inversely as the perpendicular distance from the centre of acceleration to the tangent to the path at that point.

If f is the central acceleration along OP, then the component of f normal to v is $f_n = f \sin \epsilon$, or, since from (30) $\sin \epsilon = \frac{p}{r}, f_n = f\frac{p}{r}$. But from page 53 we have seen that $f_n = \frac{v^2}{\rho}$, where ρ is the radius of curvature. Hence

$$f\frac{p}{r} = \frac{v^2}{\rho}, \quad \text{or} \quad v^2 = 2f \times \frac{1}{4}\left(2\rho\frac{p}{r}\right).$$

But $2\rho \sin \epsilon = 2\rho\frac{p}{r}$ is the length of the chord of curvature $2PB$ through the pole. (See figure page 85.)

From page 28 we have for a point moving from rest with uniform rate of change of speed a, $v^2 = 2a(s - s_1)$. Therefore, *for central acceleration the speed at any point of the path is equal to that acquired by a point moving from rest with constant rate of change of speed f through a space equal to one fourth the chord of curvature through the centre of acceleration.*

If the acceleration is central, its component perpendicular to the radius vector is zero, since the pole is the centre of acceleration, and we have from (24)

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The component along the radius vector is equal to the acceleration itself, if the pole is the centre of acceleration, and we have from equation (23), if the acceleration is *towards the centre,*

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -f, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where the (—) sign for f denotes motion towards the centre (page 50).

Equations (32) and (33) express all the conditions of central acceleration towards the pole, and therefore determine the motion.

If the acceleration is away from the pole we have $+f$ instead of $-f$ (page 50).

From (30) we have

$$\frac{d\theta^2}{dt^2} = \frac{c^2}{r^4},$$

and equation (33) becomes

$$\frac{d^2r}{dt^2} = \frac{c^2}{r^3} - f = r\omega^2 - f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

But we have seen, page 76, that $r\omega^2$ is the central acceleration for a point moving in a circle of radius r with the speed $r\omega$.

The rate of change of length of the radius vector $\frac{d^2r}{dt^2}$, we see from (34), is then the difference between the central acceleration f at any instant and the central acceleration at the same instant of a point moving in a circle of radius r with the same angular velocity.

This rate of change of velocity along the radius vector is called the

paracentric acceleration. Its integral or $\frac{dr}{dt}$ is the velocity of approach or recession along the radius vector and is called the **paracentric velocity.**

(*a*) **To find the speed at any point of the path.—Central acceleration.** If we multiply (32) by $rd\theta$ and (33) by dr and add, we have

$$dr\frac{d^2r}{dt^2} + rdr\frac{d\theta^2}{dt^2} + r^2d\theta\frac{d^2\theta}{dt^2} = -fdr, \quad . \quad . \quad . \quad . \quad . \quad (35)$$

or

$$d\left(\frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2}\right) = -2fdr.$$

Integrating, we obtain

$$\frac{dr^2 + r^2d\theta^2}{dt^2} = c_1 - 2\int fdr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where c_1 is a constant of integration.

If the law of variation of f is given in terms of r for any given case, we can perform the integration denoted by $\int fdr$.

From (18) we have

$$v^2 = \frac{ds^2}{dt^2} = \frac{dr^2 + r^2d\theta^2}{dt^2}.$$

Hence

$$v^2 = c_1 - 2\int fdr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

We have also from (30)

$$v = \frac{c}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Since in (37) the value of v depends only upon r, we see that the *speed for central acceleration at any two points of the path is independent of the path, and is the same for any points equally distant from the centre, the law of acceleration remaining the same.*

(*b*) **To find the time of describing any portion of the path.—Central acceleration.**

Substituting (31) in (36), we have

$$\frac{dr^2}{dt^2} + \frac{c^2}{r^2} = c_1 - 2\int fdr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Hence

$$t = \int_{r_1}^{r} \frac{dr}{\sqrt{c_1 - \frac{c^2}{r^2} - 2\int fdr}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

We have also from (31)

$$t = \frac{1}{c}\int_{\theta_1}^{\theta} r^2d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

from which r must be eliminated by means of the equation of the path and the integration performed in reference to θ.

(*c*) **To find the equation of the path.**

Substitute in (39) for dt^2 its value from (31), $dt^2 = \frac{r^4d\theta^2}{c^2}$, and we have

$$c^2\left(\frac{dr^2}{r^4d\theta^2} + \frac{1}{r^2}\right) = c_1 - 2\int fdr.. \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This equation may be simplified by putting $r=\frac{1}{u}$, and it then becomes

$$c^2\left(\frac{du^2}{d\theta^2}+u^2\right)=c_1+2\int f\frac{du}{u^2}. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

(*d*) To find the law of the acceleration for any given path.—Central acceleration.

Differentiating (43) with reference to $d\theta$, we have

$$f=c^2u^2\left(\frac{d^2u}{d\theta^2}+u\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Substituting in (38) for dt^2 its value from (31), $dt^2=\frac{r^4d\theta^2}{c^2}$, we have

$$f=\frac{c^2}{r^2}\left(\frac{1}{r}-\frac{d^2r}{r^2d\theta^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

which is the same as (44) if we put $\frac{1}{r}=u$.

From (30) we have $p^2=\frac{r^4d\theta^2}{v^2dt^2}$, and from (18) $v^2dt^2=dr^2+r^2d\theta^2$. Therefore

$$p^2=\frac{r^4d\theta^2}{dr^2+r^2d\theta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Substitute this in (42) and we have

$$\frac{c^2}{p^2}=c_1-2\int fdr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Differentiating with respect to p,

$$f=\frac{c^2dp}{p^3dr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The law of acceleration is given by (44) or (45) or (48).

From (33) and (35) we have

$$d^2\theta=-\frac{2\,dr\,d\theta}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

an expression which will often be found useful in reductions.

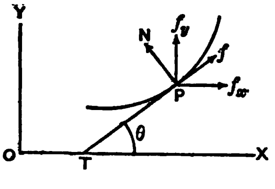
Differential Equations for Constrained Motion of a Point in a Plane.—For free motion of a point in a plane we have from (6) for the horizontal and vertical components of the acceleration

$$f_x=\frac{d^2x}{dt^2}, \quad f_y=\frac{d^2y}{dt^2}.$$

Under the action of these components, the point, if free to move, describes some curve.

But if it is constrained to move in a given curve, these components will be changed by reason of the normal acceleration N due to the given curve.

If thus f is the acceleration of a free point P, f_x and f_y its horizontal and vertical components, N the normal acceleration due to the given curve, and θ the angle of the tangent at P with the horizontal, we have



$$\frac{d^2x}{dt^2}=f_x-N\sin\theta\,; \quad . \quad . \quad . \quad (50)$$

$$\frac{d^2y}{dt^2}=f_y+N\cos\theta. \quad . \quad . \quad . \quad (51)$$

If N is zero, the motion is unconstrained and we have (6). These two equations, together with the equation of the given curve, are sufficient to determine the motion completely.

In applying them, f_x is positive towards right, negative towards left, and the horizontal component of N follows the same rule. We have f_y positive upwards and negative downwards, and the vertical component of N follows the same rule.

If we multiply (50) by $2dx$ and (51) by $2dy$ and add, we have, since $\sin\theta = \frac{dy}{ds}$ and $\cos\theta = \frac{dx}{ds}$,

$$\frac{2dxd^2x + 2dyd^2y}{dt^2} = 2(f_x dx + f_y dy).$$

The first member of this equation is the differential of $\frac{dx^2 + dy^2}{dt^2} = \frac{ds^2}{dt^2} = v^2$, or is equal to $2vdv$. Hence

$$\left.\begin{array}{ll} & vdv = f_x dx + f_y dy, \\ \text{or} & \\ & v^2 = 2\int (f_x dx + f_y dy) + \text{Constant.} \end{array}\right\} \quad . \quad . \quad . \quad (52)$$

This is precisely the same result as that obtained for free motion, equation (9).

Hence we conclude that if there is no acceleration except that of N due to the curve alone, or if $f_x = 0$, $f_y = 0$, the speed on the curve is constant and unaffected by the curve.

If there is an acceleration besides that due to the curve, the speed will be unaffected by the curve and the same as if the point were free.

If the acceleration of the free point is parallel to the axis of y, we have $f_x = 0$ and

$$v^2 = 2\int f_y dy + \text{Constant.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

If in this last case f_y is constant, we have

$$v^2 = 2f_y y + \text{Constant.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

If the distance of the point from the origin $y = s_1$ when $v = v_1$, we have

$$v^2 = v_1^2 + 2f_y(y - s_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

which is precisely the same as for uniform rate of change of speed for a free point, as given by eq. (7), page 56.

Regard must be had to the signs in applying these equations to any special case. Velocity and acceleration upwards are positive, downwards negative; to the right positive, to the left negative.

(*a*) **To find the time of motion of a point on a given curve.**

In all cases $dt = \frac{ds}{v}$. Hence when the nature of the curve and the speed at any point of it are known, the value of v may be found from (52) and substituted, and then t may be found by integration.

If the acceleration of the point is constant and equal to f and parallel to y, we have from (55)

$$dt = \frac{ds}{\sqrt{v_1^2 + 2f_y(y - s_1)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

(*b*) **To find the normal acceleration due to the curve.**

If we multiply (50) by $\frac{dy}{ds}$ and (51) by $\frac{dx}{ds}$ and subtract, we have, since $\sin\theta = \frac{dy}{ds}$, $\cos\theta = \frac{dx}{ds}$, and $dx^2 + dy^2 = ds^2$,

$$N = f_x\frac{dy}{ds} - f_y\frac{dx}{ds} + \frac{dx\,d^2y - dy\,d^2x}{ds\,dt^2}.$$

Eliminating dt by the equation $v = \frac{ds}{dt}$, we have

$$N = f_x\frac{dy}{ds} - f_y\frac{dx}{ds} + v^2\frac{dx\,d^2y - dy\,d^2}{ds^3}.$$

But if ρ is the radius of curvature of the constraining curve at the point x, y,

$$\rho = \frac{ds^3}{dx\,d^2y - dy\,d^2x}.$$

Hence

$$N = f_x\frac{dy}{ds} - f_y\frac{dx}{ds} + \frac{v^2}{\rho}$$

$$= f_x\sin\theta - f_y\cos\theta + \frac{v^2}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

The first two terms give the normal component of f for free motion. The last term is the normal acceleration due to the curve.

If f_x and f_y are zero, the only acceleration is that due to the curve and $N = \frac{v^2}{\rho}$ (page 76).

KINEMATICS OF A POINT.

TRANSLATION.

CHAPTER I.

RECTILINEAR MOTION OF TRANSLATION.

FALLING BODY. ACCELERATION INVERSELY AS THE SQUARE OF THE DISTANCE.

Translation.—We have defined translation (page 13) as motion of a rigid system, such that every straight line joining any two points remains always parallel to itself. The paths of all the points are therefore parallel at every instant and equal for any given interval of time, and the velocities of all the points at any instant are equal and parallel.

If these velocities are uniform, that is, if all points move in parallel straight lines with equal speed, the translation is **uniform.** If these velocities change either in magnitude or direction, the translation is **variable.**

When, then, a body has motion of translation only, the motion of the body is the same as that of any one of its points, and the study of the kinematics of a point is therefore the study of the translation of a body.

Rectilinear Motion.—If the direction of the acceleration of a point does not change and always coincides with the direction of the velocity, then the velocity may change in magnitude but cannot change in direction, and we have motion in a straight line.

In such case the magnitude of the velocity is the speed in a straight line, the magnitude of the acceleration is the rate of change of speed and may be either uniform or variable, and the equations of pages 28 or 51 apply.

Acceleration is Proportional to Force.—Although we are now studying change of motion without reference to its cause, it will be well for the student to keep in mind the fact that no material body can change its own motion. Any change of motion is always found to be due to the action of other bodies. This action of external bodies upon the body considered to which change of motion or acceleration is due is called **force.**

The student may figure to himself such a force as the pressure or pull of an imponderable spiral spring upon the body, the axis of the spring having always the direction of the acceleration, and the spring moving with the body, so that its pressure or pull is exerted during the entire time of acceleration and is always proportional to the acceleration.

If the acceleration changes in direction, the axis of this spring changes, so that it is always in the same direction as the acceleration.

If the acceleration changes in magnitude, the pull or push of the spring changes correspondingly.

If the acceleration is uniform, that is, does not change either in direction or magnitude, the axis of the spring does not change in direction and its pull or push is constant.

The force of gravity upon bodies near the surface of the earth is like the action of such a spring. Its action is practically constant in intensity and direction.

The student should note that the direction of the force or acceleration *is not necessarily that of the motion*, except in the case of rectilinear motion.

Thus in the case of a point moving with uniform speed in a circle, the direction of motion at any instant is tangent to the circle, but the acceleration is always directed towards the centre.

Central Acceleration.—When the acceleration is thus always directed towards or away from a fixed point, it is called **central** acceleration, and the fixed point is called the **centre of acceleration.**

If the direction of the acceleration is towards the centre, the acceleration is negative (page 50) and the force **attractive.** If away, it is positive and the force **repulsive.**

Uniform Acceleration—Motion Rectilinear—Force Attractive.—When the direction of the uniform acceleration coincides with that of the motion, we have motion in a straight line with uniform rate of change of speed, and equations (2) to (7), page 28 or 51, apply.

The most common instance of such motion is that of a body falling freely near the earth's surface.* In this case the acceleration due to gravity is known to be practically constant and is always denoted by g. We have then simply to replace a or f by g in equations (2) to (7), page 28 or 51. We shall take $g = 32.2$ ft.-per-sec. per sec. or 981 cm.-per-sec. per sec. unless otherwise specified.

Value of g.—The value of g is usually given in feet-per-sec. per sec. or in centimeters-per-sec. per sec.

It has been determined by much careful experiment and found to vary with the latitude λ and the height h above sea-level.

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* Strictly speaking there is no known instance in nature of a uniform acceleration (or of a force which does not vary in magnitude and direction). The acceleration g due to gravity (or the force of gravity) varies inversely as the square of the distance from the centre of the earth for a body outside the earth, and directly as the distance for a body inside, i.e., in a shaft or well.

But, as we shall see, the variation due to this cause is insensible for all ordinary distances. The decrease of g at a distance of a mile above the earth's surface is only about the 2000th part of its value at the surface. Also two radii of the earth are sensibly parallel when near together. It is therefore customary and practically correct to speak of g as a constant acceleration at any place.

It should be borne in mind, however, that even then the resistance of the air very materially modifies the results for falling bodies. We can therefore only assume g as constant for fall *in vacuo.*

The general value is given by

$$g = 32.173 - 0.0821 \cos 2\lambda - 0.000003h,$$

where h is the height above sea-level in feet, and g is given in feet-per-sec. per sec., or

$$g = 980.0056 - 2.5028 \cos 2\lambda - 0.000003h,$$

where h is the height above sea-level in centimeters and g is given in centimeters-per-sec. per sec.

It will be seen that the value of g increases with the latitude, and is greatest at the poles and least at the equator. It also decreases as the height above sea-level increases.

The following table gives the value of g at sea-level in a few localities:

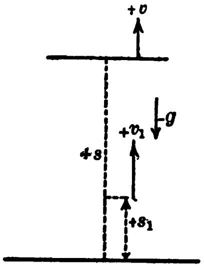
	Latitude.	g F. S. Units.	g C. S. Units.
Equator	0° 0′	32.091	978.10
New Haven	41 18	32.162	980.284
Latitude 45°	45 0	32.173	980.61
Paris	48 50	32.183	980.94
London	51 40	32.182	980.889
Greenwich	51 29	32.191	981.17
Berlin	52 30	32.194	981.25
Edinburgh	55 57	32.203	981.54
Pole	90 0	32.255	983.11
United States	49 0	32.162	980.26
	25 0	32.12	979.00

For calculations where great accuracy is not required it is customary to take $g = 32$ ft.-per-sec. per sec. or $g = 981$ cm.-per-sec. per sec.

For the United States $g = 32\frac{1}{6}$ is a good average value and is therefore very often used.

In exact calculations the value of *g for the place* must be used.

Formulas for a Body Projected Vertically Up or Down. — We have then, for a body projected vertically upwards *in vacuo*, simply to put $-g$ in place of f in equations (2) to (7), page 51. We thus obtain



$$v = v_1 - gt; \quad \ldots \quad (1)$$

$$t = \frac{v_1 - v}{g}; \quad \ldots \quad (2)$$

$$s - s_1 = \frac{v + v_1}{2}t = v_1 t - \frac{1}{2}gt^2; \quad \ldots \quad (3)$$

$$t = \frac{2(s - s_1)}{v + v_1} = \frac{v_1 \pm \sqrt{v_1^2 - 2g(s - s_1)}}{g}; \quad . \quad (4)$$

$$v^2 = v_1^2 - 2g(s - s_1); \quad \ldots \quad (5)$$

$$s - s_1 = \frac{v_1^2 - v^2}{2g}. \quad \ldots \quad (6)$$

If the starting-point is below the origin, we should change the sign of s_1.

If the body is projected downwards, we should change the signs of v, v_1, s and s_1. We see that this is equivalent to simply changing the sign of g in all equations, leaving the signs of the other quantities unchanged.

When the final velocity v is zero, we have from (2), for the time of rising to the highest point or the "*turning-point*,"

$$T = \frac{v_1}{g}.$$

For the time of rising to the highest point and returning to the starting-point we make $s - s_1 = 0$ in (4) and obtain

$$2T = \frac{2v_1}{g}.$$

Hence, *the times of rising and returning are equal.*

For body falling we have $T = -\frac{v_1}{g}$, the minus sign denoting time before the start necessary to acquire the velocity v_1.

The distance from the starting-point to the turning-point is found from (6), by making $v = 0$, to be $\frac{v_1^2}{2g}$.

The distance $\frac{v_1^2}{2g}$ or $\frac{v^2}{2g}$ is called the *height due to the velocity* v_1 or v; that is, the distance a body must fall from rest in order to acquire the velocity v_1 or v.

When the distances in rising and falling are equal we have $s - s_1 = 0$, or $\frac{v_1^2}{2g} = \frac{v^2}{2g}$, or $v_1 = v$; that is, the *velocity of return is equal to the velocity of projection.*

If the time of rising is less than $T = \frac{v_1}{g}$, the displacement $s - s_1$ is equal to the distance described. But if the time of rising is greater than $T = \frac{v_1}{g}$, the body reaches the turning-point and then falls from rest, and the entire distance described is

$$\textit{distance described} = \frac{v_1^2}{2g} + \frac{1}{2}g(t - T)^2 = \frac{v_1^2}{g} - s = \frac{v_1^2 + v^2}{2g}. \qquad (7)$$

[**Application of Calculus to the preceding Case.**]—We can deduce the preceding equations from our general equations (8) to (10), page 51.

Thus from equation (9) we have, for acceleration directed downwards and therefore (−), (page 50,)

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -g.$$

Integrating, we have

$$v = \frac{ds}{dt} = -gt + \text{Const.}$$

When $t = 0$, let $v = +v_1$, the (+) sign denoting motion upwards. Then we have Const. $= v_1$, and

$$v = \frac{ds}{dt} = v_1 - gt, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

which is equation (1), page 93.

Integrating again,

$$s = v_1t - \frac{1}{2}gt^2 + \text{Const.}$$

Let $s = +s_1$ when $t = 0$, the (+) sign denoting distance upwards.
Then we have Const. $= s_1$, and

$$s - s_1 = v_1 t - \frac{1}{2} g t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is equation (3), page 93.

From equations (1) and (3) we can deduce all the others page 93.

The student should note especially that these equations have been deduced for body projected upwards or v_1 positive.

If we suppose motion towards the centre or downwards, we should have v_1 negative.

Also if the starting-point is below the origin, we should change the sign of s_1.

In all cases we take g minus, as long as the acceleration is directed downwards.

EXAMPLES.

Unless otherwise specified $g = 32.2$ ft.-per-sec. per sec. or 981 cm.-per-sec. per sec. All bodies supposed to move in vacuum.

(1) *A point moves with a uniform velocity of* 2 *ft. per sec. Find the distance from the starting-point at the end of one hour.*

Ans. 7200 ft. Motion in a straight line.

(2) *Two trains have equal and opposite uniform velocities and each consists of* 12 *cars of* 50 *ft. They are observed to take* 18 *sec. to pass. Find their velocities.*

Ans. 22.73 miles per hour.

(3) *Two points move with uniform velocities of* 8 *and* 15 *ft. per sec. in directions inclined* 90°. *At a given instant their distance is* 10 *ft. and their relative velocity is inclined* 30° *to the line joining them. Find* (a) *their distance when nearest;* (b) *the time after the given instant at which their distance is least.*

Ans. (a) 5 ft.; (b) $\frac{5}{17}\sqrt{3}$ sec.

(4) *A body is projected vertically upwards with a velocity of* 300 *ft. per sec. Find* (a) *its velocity after* 2 *sec.;* (b) *its velocity after* 15 *sec.;* (c) *the time required for it to reach its greatest height;* (d) *the greatest height reached;* (e) *its displacement at the end of* 15 *sec.;* (f) *the space traversed by it in the first* 15 *sec.;* (g) *its displacement when its velocity is* 200 *ft. per sec. upwards;* (h) *the time required for it to attain a displacement of* 320 *ft.**

Ans. (a) 235.6 ft. per sec.; (b) 183 ft. per sec. downwards; (c) 9.3 sec.; (d) 1397.5 ft.; (e) 877.5 ft. upwards; (f) 1917.5 ft.; (g) 776.3 ft. upwards; (h) 1.13 sec. in ascending, 17.5 sec. in descending.

(5) *A ball is projected upwards from a window half way up a tower* 117.72 *meters high, with a velocity of* 39.24 *m. per sec. Find the time and speed* (a) *with which it passes the top of the tower ascending;* (b) *the same point descending;* (c) *reaches the foot of the tower.*

Ans. (a) 2 sec.; 19.62 m. per sec.; (b) 6 sec.; 19.62 m. per sec.; (c) $(4 + 2\sqrt{7})$ sec.; 19.62 $\sqrt{7}$ m. per sec.

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* If the student will refer to the Examples, page 114, he will gain an idea of the effect of the air in modifying the motion of falling bodies, and will better appreciate the delusive nature of all problems which ignore it.

(6) *A stone is dropped into a well and the splash is heard in* 3.13 *sec. If sound travels in air with a uniform velocity of* 332 *meters per sec., find the depth of the well.*

Ans. 44.1 meters.

(7) *If in the preceding example the time until the splash is heard is T and the velocity of sound in air is V, find the depth.*

$$\text{Ans. Depth} = \frac{V}{g}\left[(Tg + V) - \sqrt{V(2Tg + V)}\right]$$

$$= \left\{\left(\frac{V^2}{2g} + VT\right)^{\frac{1}{2}} - \frac{V}{(2g)^{\frac{1}{2}}}\right\}^2.$$

(8) *Show that a body projected vertically upwards requires twice as long a time to return to its initial position as to reach the highest point of its path, and has on returning to its initial position a speed equal to its initial speed.*

(9) *A stone projected vertically upwards returns to its initial position in* 6 *sec. Find* (*a*) *its height at the end of the first second, and* (*b*) *what additional speed would have kept it* 1 *sec. longer in the air.*

Ans. (*a*) 80.5 ft.; (*b*) 16.1 ft. per sec.

(10) *A body let fall near the surface of a small planet is found to traverse* 204 *ft. between the fifth and sixth seconds. Find the acceleration.*

Ans. 20.4 ft.-per-sec. per sec.

(11) *A particle describes in the nth second of its fall from rest a space equal to p times the space described in the* $(n-1)$*th second. Find the whole space described.*

Ans. $\dfrac{g(1-3p)^2}{8(1-p)^2}$.

(12) *A body uniformly accelerated, and starting without initial velocity, passes over b feet in the first p seconds. Find the time of passing over the next b ft.*

Ans. $p(\sqrt{2} - 1)$ sec.

(13) *A ball is dropped from the top of an elevator* 4.905 *meters high. Acceleration of gravity is* 9.81 *meters-per-sec. per sec. Find the times in which it will reach the floor* (*a*) *when the elevator is at rest;* (*b*) *when it is moving with a uniform downward acceleration of* 9.81 *m.-per-sec. per sec.;* (*c*) *when moving with a uniform downward acceleration of* 4.905 *m.-per-sec. per sec.;* (*d*) *when moving with a uniform upward acceleration of* 4.905 *m.-per-sec. per sec.*

Ans. (*a*) 1 sec.; (*b*) ∞; (*c*) $\sqrt{2}$ sec.; (*d*) $\sqrt{\frac{2}{3}}$ sec.

(14) *If s_1, s_2 are the heights to which a body can be projected with a given initial vertical velocity at two places on the earth's surface at which the accelerations of falling bodies are g_1 and g_2 respectively, show that $s_1g_1 = s_2g_2$.*

(15) *A stone A is let fall from the top of a tower* 483 *ft. high. At the same instant another stone B is let fall from a window* 161 *ft. below the top. How long before A will B reach the ground?*

Ans. $(\sqrt{6} - 2)\sqrt{5}$ sec.

(16) *A ball falling from the top of a tower had descended a foot when another was let fall at a point b below the top. Show that if they reach the ground together, the height of the tower is* $\frac{(a+b)^2}{4a}$ *ft.*

(17) *If two bodies are projected vertically upwards with the same initial velocity V, at an interval of t sec., prove that they will meet at a height* $\frac{g}{2}\left(\frac{V^2}{g^2} - \frac{t^2}{4}\right)$.

(18) *Two stones are falling in the same vertical line. Show that if one can overtake the other, it will do so after the same lapse of time, even if gravity ceases to act.*

(19) *Bodies are projected vertically downwards from heights* h_1, h_2, h_3 *with velocities* v_1, v_2, v_3, *and all reach the ground at the same moment. Show that*

$$\frac{h_1 - h_2}{v_1 - v_2} = \frac{h_2 - h_3}{v_2 - v_3} = \frac{h_3 - h_1}{v_3 - v_1}.$$

(20) *Two points move in straight lines with uniform accelerations. Show that if at any instant their velocities are proportional to their respective accelerations, the path of either relative to the other will be rectilinear.*

(21) *Upon the top of a tower 200 feet high is placed a flag-staff of 26 feet; a bullet is let fall from the top of this flag-staff, and at the instant of its passing the bottom of it a stone is let fall from a window 44 feet from the top of the tower. At what distance from the bottom of the tower will the bullet overtake the stone? Show also that this distance is independent of the value of g. In what time after the dropping of the stone do they meet? In what time after the dropping of the bullet? How far does the stone fall before meeting? Take acceleration due to gravity 32.16 ft.-per-sec. per sec.*

Ans. Distance $= 200 - \left[44 + \frac{44^2}{4 \times 26}\right] = 137.385$ ft.

Time from falling of stone 1.07 sec. Time from falling of bullet 2.34 sec. The stone falls 18.615 ft. The bullet falls 88.615 ft.

(22) *A body falls a distance a from rest when another body is let fall from a distance a + b below the starting-point of the first. How far will the latter body fall before it is overtaken by the former? What is the time of fall of the latter body?*

Ans. $x = \frac{b^2}{4a}$; time $= \sqrt{\frac{b^2}{2ag}}$.

(23) *A body is projected upward with a velocity which would take it to the height a, and at the same instant a body is let fall from a distance above the point of projection of b. At what distance below the latter point will the bodies meet? In what time?*

Ans. $x = \frac{b^2}{4a}$; time $= \sqrt{\frac{b^2}{2ag}}$.

(24) *A body is thrown vertically upward with a velocity* v_1. *Find the time at which it is at a given height h in its ascent.*

Ans. $t = \frac{v_1 \pm \sqrt{v_1^2 - 2gh}}{g}$.

The lower sign gives the time when the body is at the height h in ascending, the upper in descending.

(25) *A body is projected vertically upward and the interval between the times of its passing a point whose height is h in its ascent and descent is 2t. Find the velocity v_1 of projection and the whole time T of motion.*

Ans. $v_1 = \sqrt{g^2t^2 + 2gh}$, $T = \dfrac{2\sqrt{g^2t^2 + 2gh}}{g}$.

(26) *A body falling to the ground is observed to pass through eight ninths of its original height in the last second. Find the height.*

Ans. $\frac{9}{8}g = 36$ ft. nearly.

(27) *A body falling under the action of gravity is observed to describe* 144.9 *feet and* 177.1 *feet in two successive seconds. Find g and the time from the beginning of the motion to the first of the two seconds.*

Ans. $g = 32.2$ ft.-per-sec. per sec., $t = 4$ sec.

(28) *A falling body is observed at one portion of its path to pass through n feet in t sec. Find the distance described in the next t seconds.*

Ans. $n + gt^2$ feet.

(29) *A body is projected vertically upwards with a velocity 3g. At what times will its height be 4g, and what will be its velocity at these times?*

Ans. It will be at the height $4g$ at the end of 2 sec. and again at the end of 4 sec. Its velocity at both these instants is g ft. per sec. upward at the end of 2 sec. and downward at the end of 4 sec.

(30) *Find the velocity with which a body must be projected up a smooth inclined plane, the height of which is h and length l, to reach the top.*

Ans. The vertical acceleration is g. The component acceleration parallel to the plane is $a = \frac{gh}{l}$. Therefore $v = \sqrt{2al} = \sqrt{2gh}$, or the same as the velocity required to project the body to the height h.

(31) *Find the time of falling down the whole length of a smooth inclined plane of length l and height h.*

Ans. $l\sqrt{\frac{2}{gh}}$. For constant height h the time is directly as the length.

(32) *Find the speed attained by a body in falling down a smooth inclined plane the height of which is h and length l.*

Ans. $\sqrt{2gh}$, the same as in falling through the height h.

(33) *A body is projected down a smooth plane the inclination of which with the horizontal is* 45°, *with a velocity of* 10 *ft. per sec. Find the space described in* 2½ *seconds* ($g = 32$).

Ans. 95.7 ft.

(34) *A locomotive starts down a smooth incline with a velocity of* 7½ *miles an hour. If the ratio of the height to length is* $\frac{1}{200}$, *find the space traversed in two minutes* ($g = 32$).

Ans. 2472 ft.

Acceleration Inversely as the Square of the Distance from a Fixed Point—Motion Rectilinear—Force Attractive.—This is the case of a body at a great distance from the earth, under the action of the force of gravity, since in such case the acceleration is towards the centre of the earth and varies inversely as the square of the distance from the centre of the earth.*

Let s_1 be the initial distance from the centre of acceleration, the velocity at this point being v_1, and s the distance to any other position at which the velocity is v.

If the acceleration is towards the centre or force attractive, it is negative (page 50).

Let a' be the known acceleration at a distance r', and a the acceleration at any distance s. Then we have

$$a : a' :: r'^2 : s^2, \quad \text{or} \quad a = \frac{a'r'^2}{s^2}.$$

Thus for instance, in the case of the earth, a' is g, and r' is the radius of the earth at the locality where g is known.

We have then, if τ is an indefinitely small time, for acceleration towards the centre

$$\frac{v - v_1}{\tau} = a = -\frac{a'r'^2}{s^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The mean velocity for an indefinitely short time is $\frac{v + v_1}{2}$, and the distance described in this time is

$$s - s_1 = \frac{v + v_1}{2}\tau, \quad \text{or} \quad v + v_1 = \frac{2(s - s_1)}{\tau}.$$

Multiplying by (1),

$$v^2 - v_1^2 = -\frac{2a'r'^2}{s^2}(s - s_1).$$

But if the time is indefinitely small, s^2 will equal ss_1, and hence

$$v^2 - v_1^2 = -2a'r'^2\left(\frac{1}{s_1} - \frac{1}{s}\right),$$

or

$$v^2 = v_1^2 - 2a'r'^2\left(\frac{1}{s_1} - \frac{1}{s}\right). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the body falls, s_1 is greater than s and the last term becomes essentially positive. If the body is projected upwards, s_1 is less than s and the last term is essentially negative. Equation (2) holds good, then, without change in either case if the acceleration is towards the centre, or force attractive. It also holds for any path straight or curved if s, s_1, r' are ***measured along the path***, and a' is the tangential acceleration or rate of change of speed at distance r', and v and v_1 are the speeds final and initial.

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* This is the 'law of universal gravitation" as discovered by Newton. It is also known as the law of the *inverse squares*.

It is regarded as rigidly true for every particle of matter acting upon every other particle. But, as we shall see, it is not rigidly true for bodies of finite dimensions acting upon similar bodies, unless those bodies are homogeneous spheres or spherical shells.

The earth is not a sphere and is not homogeneous. Therefore it is not rigidly true that it attracts external bodies with a force inversely as the square of the distance from the centre. The deviation from this law for bodies at great distances is, however, insensible.

If the acceleration is away from the centre, or force repulsive, we should have the sign before the last term (+) instead of (—), or the sign of a' is changed in (1) and (2).

If the body falls from rest from a distance s_1, we have from (2), by making $v_1 = 0$, for a body falling from rest,

$$v^2 = 2a'r'^2\left(\frac{1}{s} - \frac{1}{s_1}\right) \textit{ falling.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the body is projected upwards and the velocity v is zero at the height s, we have from (2), for a body projected upwards to a distance s,

$$v_1^2 = 2a'r'^2\left(\frac{1}{s_1} - \frac{1}{s}\right) \textit{ rising.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

COR. 1. If the distance s_1 is infinite, we have from (3) the velocity acquired in falling from an infinite distance,

$$v = -\sqrt{\frac{2a'r'^2}{s}};$$

and from (4) the velocity of projection in order to go to an infinite distance is

$$v_1 = +\sqrt{\frac{2a'r'^2}{s_1}}.$$

In the case of a body attracted by the earth we have $a' = g$. If then in the first case $s = r'$ and in the second case $s_1 = r'$, we have for the velocity acquired in falling to the surface of the earth from an infinite distance, or the velocity necessary to project a body to an infinite distance from the surface of the earth, *in vacuo*,

$$v = v_1 = \mp\sqrt{2gr'}.$$

If we take $g = 32\frac{1}{6}$ ft.-per-sec. per sec. and the mean radius of the earth 3960 miles, we have

$$v = v_1 = \left(\frac{64\frac{1}{3} \times 3960}{5280}\right)^{\frac{1}{2}} = \mp 6.95 \text{ miles per sec.}$$

The (—) sign for falling and the (+) sign for upward projection.

COR. 2. If we put equation (3) in the form

$$v^2 = 2gr'^2\left(\frac{s_1 - s}{ss_1}\right),$$

we see at once that if $s_1 - s$ is a small distance compared to s, so that the entire fall takes place near the earth's surface, ss_1 will be practically equal to r'^2 and we shall have

$$v^2 = 2g(s_1 - s).$$

This is the same formula as for uniform acceleration g towards the earth, the initial velocity being zero (page 93).

[**Application of Calculus to the Preceding Case.**]—We can deduce the preceding results from our general equations (8) to (10), page 51.

Thus for acceleration towards the centre we have, as before,

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -\frac{a'r'^2}{s^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For acceleration away from the centre we should have (+) instead of (−) (page 50).

Multiply by ds, both sides, and then, since $\frac{ds}{dt} = v$, we have

$$vdv = -\frac{a'r'^2ds}{s^2}.$$

Integrating, we have

$$\frac{v^2}{2} = \frac{a'r'^2}{s} + \text{Const.}$$

When $s = s_1$, let $v = +v_1$ for body projected upwards and $v = -v_1$ for body projected downwards. Then in both cases Const. $= \frac{v_1^2}{2} - \frac{a'r'^2}{s_1}$, and in both cases

$$v^2 = v_1^2 - 2a'r'^2\left(\frac{1}{s_1} - \frac{1}{s}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These are the equations (1) and (2) of the preceding Article. They hold, as we see, for motion towards or away from the centre, provided the acceleration is towards the centre. For acceleration away from the centre we change the sign of a'. These equations also hold for any path, straight or curved, if s, s_1, r' are measured *along the path*, and a' is the tangential acceleration or rate of change of speed at distance r', and v and v_1 the speeds final and initial.

If the initial velocity is zero, we have for a body *falling* from rest

$$v^2 = 2a'r'^2\left(\frac{1}{s} - \frac{1}{s_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since $v = \frac{ds}{dt}$, we have from (3)

$$v = \frac{ds}{dt} = -\sqrt{2a'r'^2\left(\frac{1}{s} - \frac{1}{s_1}\right)},$$

where we take the (−) sign for the radical to denote motion *towards* the centre (page 44). This can be put in the form

$$-\frac{sds}{\sqrt{s_1s - s^2}} = dt\sqrt{\frac{2a'r'^2}{s_1}}.$$

To put this in a form convenient for integration, add and subtract $\frac{1}{2}s_1$ to the numerator of the first term. We then have

$$\frac{\frac{1}{2}s_1 - s - \frac{1}{2}s_1}{\sqrt{s_1s - s^2}}ds = \frac{s_1 - 2s}{2\sqrt{s_1s - s^2}} - \frac{s_1ds}{2\sqrt{s_1s - s^2}} = dt\sqrt{\frac{2a'r'^2}{s_1}}.$$

Integrating, we have

$$(s_1s - s^2)^{\frac{1}{2}} - \frac{s_1}{2}\text{versin}^{-1}\frac{2s}{s_1} = \left(\frac{2a'r'^2}{s_1}\right)^{\frac{1}{2}}t + \text{Const.} \quad . \quad . \quad . \quad (4)$$

Let $t = 0$ when $s = s_1$, then Const. $= -\frac{\pi s_1}{2}$. Hence* for the time of falling from rest we have for acceleration towards the centre

---

* We have $\pi - \text{versin}^{-1}\frac{2s}{s_1} = \pi - \cos^{-1}\left(1 - \frac{2s}{s_1}\right) = \cos^{-1}\left(\frac{2s}{s_1} - 1\right)$.

From trigonometry, $2\cos^2 y - 1 = \cos 2y$. Let $2y = \cos^{-1}\left(\frac{2s}{s_1} - 1\right)$. Then $\cos 2y = \frac{2s}{s_1} - 1$ and $\cos^2 y = \frac{s}{s_1}$, or $y = \cos^{-1}\sqrt{\frac{s}{s_1}}$ and $2y = 2\cos^{-1}\sqrt{\frac{s}{s_1}}$.

Hence $2\cos^{-1}\sqrt{\frac{s}{s_1}} = \pi - \text{versin}^{-1}\frac{2s}{s_1}$.

$$t = \left(\frac{s_1}{2a'r'^2}\right)^{\frac{1}{2}} \left\{ (s_1 s - s^2)^{\frac{1}{2}} + s_1 \cos^{-1}\left(\frac{s}{s_1}\right)^{\frac{1}{2}} \right\}. \quad . \quad . \quad . \quad . \quad (5)$$

If we had taken motion *away* from the centre, we should have obtained, instead of (4),

$$-(s_1 s - s^2)^{\frac{1}{2}} + \frac{s_1}{2} \operatorname{versin}^{-1} \frac{2s}{s_1} = \left(\frac{2a'r'^2}{s_1}\right)^{\frac{1}{2}} t + \text{Const.}$$

Let $t = 0$, when $s = 0$, and Const. $= 0$, and we have

$$t = \left(\frac{s_1}{2a'r'^2}\right)^{\frac{1}{2}} \left\{ \frac{s_1}{2} \operatorname{versin}^{-1} \frac{2s}{s_1} - (s_1 s - s^2)^{\frac{1}{2}} \right\}. \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) applies to a body projected upwards to any height s, s_1 being the height at which it would come to rest.

If we make $s = s_1$ in (6), or $s = 0$ in (5), we find the time of reaching the turning-point in rising or reaching the centre in falling from rest

$$T = \frac{\pi s_1 \sqrt{s_1}}{2\sqrt{2a'r'^2}}.$$

CHAPTER II.

SIMPLE HARMONIC MOTION. MOTION IN RESISTING MEDIUM.

Simple Harmonic Motion.—The motion of a point moving in any path in such a manner that the tangential acceleration is directly proportional to the distance, along the path, from a fixed point in the path is called **simple harmonic motion.** Such motion may be rectilinear or curvilinear.

The vibrations of such bodies as a tuning-fork or a piano-wire are approximate examples of such motion, and hence the term "harmonic." The vibrations of an elastic body, such as the air, are examples of such motion.

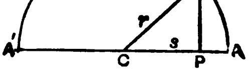
It is also, as has been stated (note, page 92), the motion of a body under the action of gravitation, within a homogeneous sphere, as it can be shown that in this case the acceleration due to gravity is proportional to the distance from the centre.

The motion of the piston of a steam-engine when moved by a crank and connecting-rod approximates the same motion if the rotation of the crank is uniform, the approximation being closer the longer the connecting-rod. This will be evident from the following Article.

Simple Harmonic Motion in a Straight Line—Force Attractive.—Let a point M move with uniform speed in a circle of radius $CM = r$.

Then the acceleration f_n is always directed towards the centre and equal to $f_n = r\omega^2$, where ω is the constant angular velocity (page 76).

The projection of f_n upon the diameter CA is $\omega^2 r \cos MCP$. But $r \cos MCP$ is the distance $CP = s$ of the projection P of M upon the diameter CA. Therefore the projection of f_n upon the diameter is $a = \omega^2 s$, or, since ω is constant, a is directly proportional to the distance $CP = s$. The motion of P is therefore harmonic.



If then a point M moves with uniform speed in a circle, *its projection P upon any diameter moves with harmonic motion in the diameter, the centre of acceleration being the centre of the circle.*

Let a' be the known acceleration in the line AC of P at a given distance r' from the centre. Then $a' = \omega^2 r'$ and $\omega = \sqrt{\frac{a'}{r'}}$, and the speed of M is $r\omega = r\sqrt{\frac{a'}{r'}}$. The projection of this speed on the diameter is $\sqrt{\frac{a'}{r'}}\, r \cos CMP = MP\sqrt{\frac{a'}{r'}}$.

But $MP = \sqrt{r^2 - s^2}$; hence we have for the velocity of the point P in the line AC at the distance $s = CP$ from C,

$$v^2 = \frac{a'}{r'}(r^2 - s^2), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

or, if v_1 is the initial velocity,

$$v^2 = v_1^2 + \frac{a'}{r'}(r^2 - s^2),$$

where a' is the known acceleration of P at a given distance r' from C.

Thus the point P starts from rest at the distance $s = r$ from C. The velocity increases as the distance s decreases, till P arrives at the centre C where the velocity is a maximum and equal to $v = r\sqrt{\frac{a'}{r'}}$. Then the velocity decreases and finally becomes zero when P arrives at A' at the distance $s = -r$ on the other side of C. Equation (1) holds good for motion towards or away from the centre if the acceleration is towards the centre, or force attractive. It also holds for any path straight or curved if s, r, r' are measured along the path and a' is the tangential acceleration or rate of change of speed at distance r', and v and v' the speeds final and initial.

COR. 1. Since the uniform speed of M in the circle is $r\sqrt{\frac{a'}{r'}}$, the time occupied by P in passing from A to A' and back to A is

$$T = \frac{2\pi r}{r\sqrt{\frac{a'}{r'}}} = 2\pi\sqrt{\frac{r'}{a'}} = \frac{2\pi}{\omega}.$$

But if a is the acceleration at any distance s, and f_n is the acceleration at the extreme distance r, we have for harmonic motion

$$\frac{r'}{a'} = \frac{r}{f_n} = \frac{s}{a} = \frac{1}{\omega^2}.$$

Hence the time T of a complete oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{s}{a}}.$$

The time T of a complete oscillation depends therefore only upon the constant ratio $\frac{s}{a} = \frac{1}{\omega^2}$ and is independent of the range r or amplitude of the oscillation. For this reason the oscillations are said to be isochronous, or made in equal times, no matter what the range or amplitude.

COR. 2. Since the motion of a body under the action of gravity in a homogeneous sphere is harmonic (page 92), if we put g for a' and let r' be the mean radius of the earth, we have from (1) the motion of a body falling under the action of gravity towards the centre of the earth in a well or shaft, assuming the earth to be a homogeneous sphere and neglecting resistance of the air.

In such case (1) becomes

$$v^2 = \frac{g}{r'}(r + s)(r - s).$$

If the fall takes place for a short distance compared to r' and near the surface, we have $r + s$ practically equal to $2r'$ and hence

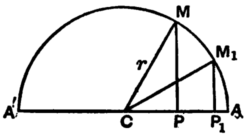
$$v^2 = 2g(r - s),$$

which is the same as for uniform acceleration g, the initial speed being zero.

We obtained the same result (page 93) for a body external to the earth. The equations of page 93 hold good, therefore, in all practical cases, whether the fall takes place above the earth or within the earth, neglecting resistance of the air.

Amplitude—Epoch—Period—Phase.—The range $r = CA = CA'$ on either side of the centre of acceleration, in harmonic motion, is called the **amplitude.**

A complete oscillation is from A to A' and back to A. The time of an oscillation, as we have seen, is independent of the amplitude. From A to A' or A' to A is a **vibration.** A vibration is half an oscillation. The time of a vibration is half that of a complete oscillation.



If P_1 is the initial position from which the time is counted, or the position of P at zero of time, the time of passing from A to P_1 is called the **epoch.** The epoch may also be defined with reference to the auxiliary circle, as the angle ACM_1 in radians. This is the epoch in *angular measure.*

The epoch in angular measure is then the angle described on the auxiliary circle in the interval of time defined as the epoch.

The epoch locates the position of P at zero of time.

The entire time which elapses from any instant until the moving point again moves in the same direction through the same position is called the **period.** The time from P_1 to A', then back through P_1 to A, and finally back from A to P_1, is a period. It is evidently the time of a complete oscillation from A back to A.

That fraction of the period which has elapsed since the moving point P last occupied A is called the **phase.** Measured on the circle, it is the ratio of the angle ACM radians to 2π radians.

The phase locates the position of P at any instant.

It therefore varies with the time or with the position of P. The phase at zero of time, then, multiplied by 2π radians gives the epoch in angular measure, and multiplied by the time of an oscillation gives the epoch in time.

[**Application of Calculus to Harmonic Motion.**—We may deduce the results obtained for simple harmonic motion (page 104), as well as others, from the general equations (8) to (10), page 51.

We have, as before, $a = \frac{a'}{r'}s$ (page 104). For acceleration away from the centre we have a positive, for acceleration towards the centre a negative (page 50).

I. **Acceleration towards the Centre.**—In this case we have

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -\frac{a'}{r'}s.$$

Multiply both sides by ds and then, since $\frac{ds}{dt} = v$, we have

$$vdv = -\frac{a'}{r'}sds.$$

Integrating, we obtain

$$v^2 = -\frac{a's^2}{r'} + \text{Const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

When $s = r$ let $v = v_1$. We have then

$$\text{Const.} = v_1^2 + \frac{a'r^2}{r'},$$

and hence

$$v^2 = v_1^2 + \frac{a'}{r'}(r^2 - s^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the initial velocity v_1 is zero, this becomes

$$v^2 = \frac{a'}{r'}(r^2 - s^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the same as equation (1), page 104, already obtained, for initial velocity zero and range r.

Since $v = \frac{ds}{dt}$, we have from (1)

$$v = \frac{ds}{dt} = \pm\sqrt{v_1^2 + \frac{a'}{r'}(r^2 - s^2)},$$

where we take the (+) sign for motion away from the centre and the (−) sign for motion towards the centre (page 44).

This can be written

$$\frac{ds}{\sqrt{r^2 + \frac{r'}{a'}v_1^2 - s^2}} = \pm\sqrt{\frac{a'}{r'}}dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we integrate this between the limits of t and $t = 0$ when $s = r$, we have

$$\sin^{-1}\frac{s}{\sqrt{r^2 + \frac{r'}{a'}v_1^2}} = \pm\, t\sqrt{\frac{a'}{r'}} + \sin^{-1}\frac{r}{\sqrt{r^2 + \frac{r'}{a'}v_1^2}}.$$

Hence*

$$s = r\cos t\sqrt{\frac{a'}{r'}} \pm v_1\sqrt{\frac{r'}{a'}}\sin t\sqrt{\frac{a'}{r'}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

---

* Let $A = \pm\, t\sqrt{\frac{a'}{r'}}$, $B = \sin^{-1}\frac{r}{\sqrt{r^2 + \frac{r'}{a'}v_1^2}}$.

Then $\frac{s}{\sqrt{r^2 + \frac{r'}{a'}v_1^2}} = \sin(A + B) = \sin A\cos B + \cos A\sin B.$

But $\sin B = \frac{r}{\sqrt{r^2 + \frac{r'}{a'}v_1^2}}$, and $\cos B = \sqrt{1 - \sin^2 B}$. Substituting these values and reducing, we obtain equation (4).

If motion is towards the centre, we take the (—) sign; if away from the centre, the (+) sign.

If v_1 is zero, or there is no initial velocity, we have

$$\left.\begin{aligned} s &= r\cos t\sqrt{\frac{a'}{r'}}; \\ t &= \sqrt{\frac{r'}{a'}}\cos^{-1}\frac{s}{r}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

If we make $v = 0$ in (1), we have for the amplitude

$$s = R = \pm\sqrt{r^2 + \frac{r'}{a'}v_1^2}.$$

This reduces to $\pm r$ when $v_1 = 0$.

If we integrate (3) between the limits of t and $t = 0$ when $s = R$, that is, if we count the time from the end of the amplitude where $v_1 = 0$, instead of from $s = r$, we obtain

$$\sin^{-1}\frac{s}{R} = \pm t\sqrt{\frac{a'}{r'}} + \frac{\pi}{2}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and hence

$$\left.\begin{aligned} s &= R\cos t\sqrt{\frac{a'}{r'}}; \\ t &= \sqrt{\frac{r'}{a'}}\cos^{-1}\frac{s}{R}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

If in (5) we make $s = -r$ or in (7) make $s = -R$, we have in both cases for the time of a vibration, $\pi\sqrt{\frac{r'}{a'}}$, and hence for the time of a complete oscillation $T = 2\pi\sqrt{\frac{r'}{a'}}$. Therefore *the time of oscillation or vibration is not affected by the initial velocity.*

All these equations (1) to (7) hold for motion either towards or away from the centre, provided the acceleration is towards the centre.

They also hold for any path, straight or curved, provided r', r and s are measured along the path and a' is the rate of change of speed at the distance r'.

II. **Acceleration Away from the Centre.**—In this case we have the acceleration positive and hence

$$a = \frac{dv}{dt} = +\frac{a'}{r'}s.$$

Multiplying by ds, we have, since $\frac{ds}{dt} = v$,

$$vdv = \frac{a'}{r'}sds.$$

Integrating this, we have

$$v^2 = \frac{a's^2}{r'} + \text{Const.}$$

When $s = 0$, let $v = v_1$. Then Const. $= v_1^2$ and

$$v^2 = v_1^2 + \frac{a'}{r'}s^2. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

From (8) we have

$$v = \frac{ds}{dt} = \pm\sqrt{v_1^2 + \frac{a'}{r'}s^2},$$

where we take the (+) sign for motion away and the (−) sign for motion towards the centre. We can put this in the form

$$\frac{ds}{\sqrt{s^2 + \frac{r'}{a'}v_1^2}} = \pm \sqrt{\frac{a'}{r'}}dt.$$

If we integrate this between the limits of t and $t = 0$ when $s = 0$, we have

$$\operatorname{logn}\left(s + \sqrt{s^2 + \frac{r'}{a'}v_1^2}\right) = \pm\, t\sqrt{\frac{a'}{r'}} + \operatorname{logn}\; v_1\sqrt{\frac{r'}{a'}}.$$

Hence

$$s = \frac{v_1}{2}\sqrt{\frac{r'}{a'}}\left(e^{\pm t\sqrt{\frac{a'}{r'}}} - \frac{1}{e^{\pm t\sqrt{\frac{a'}{r'}}}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where e is the base of the Naperian system of logarithms.

EXAMPLES.

$g = 32.16$ ft.-per-sec. per sec. or 980.23 cm.-per-sec. per sec.

(1) *If the radius of the earth is* 6370900 *meters and the acceleration of gravity* 9.81 *meters-per-sec. per sec., what should be the value of* $a'r'^2$ *in eq.* (2), *page* 99, *if* s *and* s_1 *are given in kilometers?*

Ans. 398171.88 cubic kilometers-per-sec. per sec.

(2) *A body falls to the earth from a point* 1000 *miles above the surface. Find its speed on reaching the surface, neglecting resistance of the air and taking the earth's radius* 4000 *miles.*

Ans. $v = 3.12$ miles per sec.

(3) *In the last example find the distance from the earth's surface when the speed is* 2 *miles per sec.*

Ans. 535.2 miles.

(4) *With what speed must a body be projected vertically at the earth's surface so that it may never return?* (*Assume the earth to have no atmosphere and not to be rotating.*)

Ans. The speed is the same as that which a falling body would have falling from an infinite distance, or $v = 6.95$ miles per sec.

(5) *At what point on a line joining the centres of the earth and moon would the rate of change of speed of a body be zero?* (*At the moon's surface* $a = 5.5$ *ft.-per-sec. per sec.; radius of moon* 1080 *miles; distance between centres of earth and moon* 240000 *miles.*)

Ans. Let x = distance of point from earth's centre and x_1 from moon's centre, and R earth's radius, r moon's radius. Then $x + x_1 = 240000$, and $\frac{ar^2}{x_1^2} = \frac{gR^2}{x^2}$. Hence $x = 215893$ miles.

(6) *A point whose motion is simple harmonic has velocities* 20 *and* 25 *ft. per sec. at distances* 10 *and* 8 *ft. from the centre of acceleration. Find* (*a*) *its period,* (*b*) *its acceleration at unit distance from centre.*

Ans. We have $625 = \frac{a'}{r'}(r^2 - 64)$ and $400 = \frac{a'}{r'}(r_2^2 - 100)$. Therefore $\sqrt{\frac{a'}{r'}} = \frac{15}{6}$ and period $= \frac{2\pi}{\sqrt{\frac{a'}{r'}}} = \frac{4\pi}{5}$ sec. $a = -\frac{a's}{r'} = -\frac{225}{36} = -6.25$ ft.-per-sec. per sec.

(7) *The period of a simple harmonic motion is* 20 *sec. and the maximum velocity is* 10 *ft. per sec. Find the velocity at a distance of* $\frac{60}{\pi}$ *ft. from the mean position.*

Ans. $\frac{2\pi}{\sqrt{\frac{a'}{r'}}} = 20$ sec., therefore $\frac{a'}{r'} = \frac{\pi^2}{100}$. Where $s = 0$, $r_1 = \frac{100}{\pi}$ft.

Hence $v^2 = \frac{\pi^2}{100}\left(\frac{100^2}{\pi^2} - \frac{60^2}{\pi^2}\right)$ or $v = 8$ ft. per sec.

(8) *A point moves from rest towards a fixed point* 10 *meters distant, its acceleration being everywhere* 4 *times its distance from a fixed point. At what distance will it have a velocity of* 12 *meters per sec.?*

Ans. 8 meters.

(9) *Find the mean speed of a point executing a simple harmonic motion during the time occupied in moving from one to the other extremity of its range, its maximum speed being* 5 *ft. per sec.*

Ans. The distance is $2r$. The time $\frac{\pi}{\sqrt{\frac{a'}{r'}}}$. The mean speed $\frac{2r\sqrt{a'}}{\pi\sqrt{r'}}$.

When $s = 0$, we have $25 = \frac{a'}{r'}r^2$, or $r\sqrt{\frac{a'}{r'}} = 5$. Therefore mean speed is $\frac{10}{\pi}$ ft. per sec.

(10) *If T be the period and a the amplitude of a simple harmonic motion, and if v be the velocity and s the distance from the centre at a given instant, show that*

$$a = \left(\frac{T^2v^2}{4\pi^2} + s^2\right)^{\frac{1}{2}}.$$

(11) *A point oscillates about a centre, its acceleration being proportional to its distance. Show that the ratio of its maximum velocity to the square root of the excess of the square of its maximum velocity over the square of the velocity which it has when at a given displacement from the centre is equal to the ratio of its maximum displacement to the given displacement.*

(12) *A point has a simple harmonic motion whose period is* 4 *min.* 12 *sec. Find the time during which its phase changes from* $\frac{1}{12}$ *to* $\frac{1}{6}$ *of a period.*

Ans. 21 sec.

[Body Projected in a Resisting Medium—Acceleration Proportional to the Square of the Velocity—Motion Rectilinear.]—When a body moves in a resisting medium such as air or water it loses velocity or has a minus acceleration, which is usually assumed to vary as the square of the velocity.

We have then

$$a = \frac{dv}{dt} = -cv^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where c is a constant depending upon the shape and dimensions of the body and the density, or mass of a unit of volume, of the body and medium.

This constant is called the *coefficient of resistance.* For instance, for a sphere, if d is the diameter and Δ the density of the medium and δ the density of the body, we have, as is proved in Vol. II, Statics,

$$c = \frac{\Delta}{\delta} \cdot \frac{3}{8d}.$$

We can put (1) in the form

$$cdt = -\frac{dv}{v^2}.$$

Let $v = v_1$ when $t = 0$ and integrate, and we have

$$ct = \left(\frac{1}{v} - \frac{1}{v_1}\right), \quad \text{or} \quad t = \frac{1}{c}\left(\frac{1}{v} - \frac{1}{v_1}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$v = \frac{v_1}{1 + cv_1 t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3) we can find the time for any velocity or the reverse. Since $v = \frac{ds}{dt}$, we have from (3)

$$ds = \frac{v_1 dt}{1 + cv_1 t}.$$

Integrating and making $s = 0$ when $t = 0$, we have

$$s = \frac{1}{c} \operatorname{logn} (1 + cv_1 t); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or using common logarithms,

$$s = \frac{2.302585}{c} \log (1 + cv_1 t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (4) we have

$$t = \frac{e^{cs} - 1}{cv_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $e = 2.718282$ = base of the Naperian system of logarithms.

From (5) and (6) we can find the distance for any time or the reverse.

From (6) we have $cv_1 t = e^{cs} - 1$, and substituting this in (3) we have

$$v = \frac{v_1}{e^{cs}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$s = \frac{1}{c} \operatorname{logn} \frac{v_1}{v}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or using common logarithms

$$s = \frac{2.302585}{c} \log \frac{v_1}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From (7) and (9) we can find the velocity for any distance or the reverse.

From (8) we see that when the velocity becomes zero the space traversed is infinite, and from (2) the time is infinite. Although then the velocity diminishes as the time increases, as we see from (3), it cannot become zero in any finite distance or time.

[Body Falling under the Action of Gravity in a Resisting Medium.]—Let f be the uniform acceleration due to any attractive force. In the case of gravity we have*

$$f = g\left(1 - \frac{\Delta}{\delta}\right),$$

where Δ is the density or mass of a unit volume of the medium and δ is the density of the body. The acceleration f acts away from the starting-point and the retardation cv^2 acts towards the starting-point. If then we take this point as origin, we have

$$\frac{dv}{dt} = f - cv^2, \quad \dots\dots\dots\dots \quad (1)$$

where c is the coefficient of resistance and has the same value as in the preceding Article.

Let k be that velocity for which the retardation is equal to f, so that $f = ck^2$. Then $c = \frac{f}{k^2}$, and equation (1) becomes

$$\frac{dv}{dt} = f - \frac{f}{k^2}v^2. \quad \dots\dots\dots\dots \quad (2)$$

We can write this in the form

$$dt = \frac{k^2}{f}\frac{dv}{k^2 - v^2}. \quad \dots\dots\dots\dots \quad (3)$$

Integrating, we have

$$t = \frac{k}{2f}\operatorname{logn}\frac{k+v}{k-v} + \text{Const.}$$

When $t = 0$, let $v = v_1$. Then Const. $= -\frac{k}{2f}\operatorname{logn}\frac{k+v_1}{k-v_1}$. Hence

$$t = \frac{k}{2f}\operatorname{logn}\frac{(k+v)(k-v_1)}{(k-v)(k+v_1)}; \quad \dots\dots\dots \quad (4)$$

or using common logarithms,

$$t = \frac{2.302585k}{2f}\log\frac{(k+v)(k-v_1)}{(k-v)(k+v_1)}. \quad \dots\dots\dots \quad (5)$$

From (4) we have, if $e = 2.718282$ = base of Naperian system of logarithms,

$$v = \frac{k\left[\frac{k+v_1}{k-v_1}e^{\frac{2ft}{k}} - 1\right]}{\frac{k+v_1}{k-v_1}e^{\frac{2ft}{k}} + 1}. \quad \dots\dots\dots \quad (6)$$

From (5) and (6) we can find the time for any given velocity or the reverse.

---

* As we shall see hereafter, the mass of a body multiplied by g gives the weight of the body, that is, the force of gravity. It is also a well-known fact that a body immersed in a fluid has its weight diminished by the weight of an equal volume of the medium.

If then V is the volume of the body, $V\delta g$ is its weight in vacuo and $V\Delta g$ is its loss of weight due to the medium. Hence $V\delta g - V\Delta g$ is the weight when immersed. Since $V\delta$ is the mass,

$$V\delta f = V\delta g - V\Delta g, \quad \text{or} \quad f = g\left(1 - \frac{\Delta}{\delta}\right).$$

Since $v = \frac{ds}{dt}$, we have

$$\frac{dv}{dt} = \frac{vdv}{ds},$$

and therefore from (2) we have

$$s = \frac{k^2}{2f} \operatorname{logn} \frac{k^2 - v_1^2}{k^2 - v^2}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

or using common logarithms,

$$s = \frac{2.302585k^2}{2f} \log \frac{k^2 - v_1^2}{k^2 - v^2}. \quad . \; . \; . \; . \; . \; . \quad (8)$$

From (7) we have

$$v^2 = k^2 - (k^2 - v_1^2)e^{-\frac{2fs}{k^2}}, \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

where $e = 2.718282$ = base of the Naperian system of logarithms.

From (8) and (9) we can find the distance for any velocity or the reverse.

We see from (9) that when s is great, v approaches k, and k is the limiting value of v. If the initial velocity is zero or less than k, v will continually approach k, but can never exceed k. If the initial velocity is greater than k, v will diminish continually down to k and can never become less than k.

[Body Projected Upwards under the Action of Gravity in a Resisting Medium.]

In this case we have, taking k as before, since f is negative and the resistance is negative,

$$a = \frac{dv}{dt} = -f - \frac{f}{k^2}v^2, \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

which can be written $dt = -\frac{k^2}{f}\frac{dv}{k^2 + v^2}$.

Integrating and determining the constant by the condition that when $t = 0$, $v = v_1$ = initial velocity, we have

$$t = \frac{k}{f} \tan^{-1} \frac{k(v_1 - v)}{k^2 + vv_1}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

We have also as before $\frac{dv}{dk} = \frac{vdv}{ds}$, and therefore, from (1),

$$ds = -\frac{k^2}{f}\frac{vdv}{k^2 + v^2}.$$

Integrating this, and making $s = 0$, when $v = v_1$, we have

$$s = \frac{k^2}{2f} \operatorname{logn} \frac{k^2 + v_1^2}{k^2 + v^2}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

or in common logarithms,

$$s = \frac{2.302585k^2}{2f} \log \frac{k^2 + v_1^2}{k^2 + v^2}. \quad . \; . \; . \; . \; . \; . \quad (4)$$

The time in which the velocity becomes zero and the body reaches the turning-point is, from (2),

$$T = \frac{k}{f} \tan^{-1} \frac{v_1}{k}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

and the corresponding value of s is, from (4),

$$h = \frac{2.302585k^2}{f} \log \left(1 + \frac{v_1^2}{k^2}\right). \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

At the end of the time T the body begins to return and falls from a state of rest, or $v_1 = 0$.

We have then from the preceding Article, making $v_1 = 0$,

$$t - T = \frac{2.302585k}{2f} \log \frac{k+v}{k-v}, \quad \ldots\ldots\ldots \quad (7)$$

and

$$h - s = \frac{2.302585k^2}{2f} \log \frac{k^2}{k^2 - v^2}. \quad \ldots\ldots\ldots \quad (8)$$

Let u be the velocity with which the body returns to the starting-point. Then putting $s = 0$ in (8) and $v = u$, we have

$$\frac{u^2}{k^2} = 1 - e^{\frac{-2fh}{k^2}};$$

or substituting for h its value,

$$\frac{u^2}{k^2} = \frac{\frac{v_1^2}{k^2}}{1 + \frac{v_1^2}{k^2}}. \quad \ldots\ldots\ldots\ldots \quad (9)$$

Hence

$$\frac{1}{u^2} - \frac{1}{v_1^2} = \frac{1}{k^2}. \quad \ldots\ldots\ldots\ldots \quad (10)$$

We see then that u is less than v_1, or the body returns to the point of projection with a velocity less than the velocity of projection.

Values of $\frac{\delta}{\Delta}$ and c.—For motion in a resisting medium, under the action of gravity, we have $f = g\left(1 - \frac{\Delta}{\delta}\right)$, where Δ is the density of the medium and δ that of the body.

For iron in water we may take $\frac{\delta}{\Delta} =$ 7.2.

" " " air " " " $\frac{\delta}{\Delta} = 5983.28$.

" mist or rain in air " " " $\frac{\delta}{\Delta} = 813.82$.

" lead in water " " " $\frac{\delta}{\Delta} = 11.35$.

" " " air " " " $\frac{\delta}{\Delta} = 9423.61$.

The coefficient of resistance for a sphere (Vol. II, Statics) is

$$c = \frac{f}{k^2} = \frac{3\Delta}{16\delta r},$$

where r is the radius of the sphere.

For a cone we have

$$c = \frac{3\Delta r^2}{2\delta h(r^2 + h^2)},$$

where r is the radius of the base and h the height. If the cone terminates in a cylinder of length l, we have

$$c = \frac{3\delta r^2}{2\delta(3l + h)(r^2 + h^2)}.$$

EXAMPLES.*

$g = 32.16$ ft.-per-sec. per sec.

(1) *A lead bullet* 1 *inch in diameter is projected vertically with a velocity of* 2000 *ft. per sec. Find* (*a*) *the time of ascent with and without resistance of the air;* (*b*) *the distance to which it ascends with and without resistance of the air;* (*c*) *the velocity and time of return with and without resistance of the air.*

Ans. We have in this case $\frac{\delta}{\Delta} = 9423.61$, $k^2 = 67338.1852$ and $k = 259.49$ ft. per sec., $f = 32.1556$ ft.-per-sec. per sec.

(*a*) The time of ascent in vacuo is 62.19 sec.

$$\text{In air} \quad T = \frac{259.49}{32.1556} \tan^{-1} \frac{2000}{259.49} = 12.65 \text{ sec.}$$

(*b*) The height of ascent in vacuo is 62189 ft.

$$\text{In air} \quad h = \frac{2.302585 \times 67338.1852}{32.1556} \log \left(1 + \frac{4000000}{67338.1852}\right) = 8583 \text{ ft.}$$

(*c*) The velocity of return in vacuo is 2000 ft. per sec.

$$\text{In air} \quad \mu^2 = \frac{4000000}{1 + \frac{4000000}{67338.1852}}, \text{ or } u = 257 \text{ ft. per sec.}$$

(*d*) The time of return in vacuo is 62.19 sec.

$$\text{In air } t - T = \frac{2.302585 \times 259.49}{2 \times 32.1556} \log \frac{259.49 + 257}{259.49 - 257} = 21.52 \text{ sec.}$$

(2) *A lead bullet* 1 *inch in diameter is let fall in the air. Find the velocity at the end of* $t = 1$ *sec.,* 2 *sec.,* 3 *sec.,* 10 *sec.,* 20 *sec., with and without the resistance of the air.*

Ans. We have $k = 259.49$ ft. per sec.; $f = 32.1556$ ft.-per-sec. per sec.; $e = 2.718282$; and from eq. (6), page 111, making $v_1 = 0$, $v = \frac{k\left(e^{\frac{2ft}{k}} - 1\right)}{e^{\frac{2ft}{k}} + 1}$.

Let $t = 1, 2, 3, 10$ and 20, and we have $v = 31.98, 63.03, 92.33, 219.3$ and 255.86 ft. per sec.; while in vacuo we would have $v = 32.16, 64.32, 96.48$, 321.6 and 643.2 ft. per sec.

(3) *In the previous example what is the greatest velocity the bullet can attain?*

Ans. $k = 259.49$ ft. per sec. As we have seen, this velocity is attained quite early, after which the velocity is uniform.

(4) *An iron cannon-ball* 1 *ft. in diameter is projected vertically upwards in the air with a velocity of* 2000 *ft. per sec. Find* (*a*) *the time of ascent;* (*b*) *the distance to which it ascends;* (*c*) *the velocity with which it returns;* (*d*) *the time of return.*

Ans. We have in this case $\frac{\delta}{\Delta} = 5983.28$, $k^2 = 513040$, $k = 716.268$ ft. per sec., $f = 32.1546$ ft.-per-sec. per sec.

---

* An examination of these problems will give the student an idea of the effect of the air in modifying the motion of a falling body and enable him to realize the inaccuracy of neglecting it.

(*a*) The time of ascent is $T = 27.82$ sec.; in vacuo, 62.19 sec. (See Ex. (1).)
(*b*) The distance of ascent is $h = 84680$ ft.; in vacuo, 62189 ft.
(*c*) The velocity of return is $u = 674.3$ ft. per sec.; in vacuo, 2000 ft. per sec.
(*d*) The time of return is $t - T = 38.9$ sec.; in vacuo, 62.19 sec.

(5) *An iron cannon-ball* 1 *ft. in diameter is let fall in the air. Find the velocity at the end of* $t = 1$ *sec.*, 2 *sec.*, 3 *sec.*, 10 *sec.*, 40 *sec.*, 60 *sec.*

Ans. We have $k = 716.268$ ft. per sec., $f = 32.1546$ ft.-per-sec. per sec. and $v_1 = 0$. Hence from eq. (6), page 111:

For $t =$	1	2	3	10	40	60 sec.
$v =$	32.12	64	95.85	301.5	677.8	709.7 ft.-per-sec.
In vacuo we would have						
$v =$	32.16	64.32	96.48	321.6	1286.4	1929.6 ft. per sec.

(6) *In the previous example, what is the greatest velocity the cannon-ball can attain?*

Ans. $k = 716.268$ ft. per sec. And this is attained in little more than a minute.

(7) *In Example* (5) *what are the distances passed through?*

Ans. From eq. (8) we have, making $v_1 = 0$ and taking the values of v already found,

For $t =$	1	2	3	10	40	60 sec.
$s =$	16.03	64.29	144.2	1561	18000	31992 ft.
In vacuo we would have						
$s =$	16.08	64.32	144.72	1608	25728	57888 ft.

(8) *A lead shot* $\frac{1}{8}$ *inch in diameter is let fall in the air. Find the greatest velocity it can attain, and the velocity and space traversed in* $t = 1, 2, 3, 4, 5, 6, 7$ *and* 8 *sec.*

Ans. We have $\frac{\delta}{\Delta} = 9423.61$, $f = 32.1556$, $k^2 = 8417$.

Greatest velocity $= k = 91.7$ ft. per sec.

For $t =$	1	2	3	4	5	6	7	8 sec.
$v =$	30.89	55.49	71.76	81.23	86.35	89.01	90.35	91.03 ft. per sec.
In vacuo,								
$v =$	32.16	64.32	96.48	128.64	160.8	192.96	225.12	257.28 ft. per sec.
$s =$	15.67	60.27	123.56	199	283.3	370.7	458.1	545.5 ft.
In vacuo,								
	16.08	64.32	145.72	257.3	402	578.8	787.9	1029.12 ft.

(9) *What is the greatest velocity a rain-drop* $\frac{1}{8}$ *inch in diameter can acquire, falling in the air?*

Ans. We have $\frac{\delta}{\Delta} = 813.82$, $f = 32\ 12$, $k = 27$ ft. per sec.

(10) *An iron cannon-ball* 1 *ft. in diameter is let fall in water. Find* (*a*) *the greatest velocity it can attain;* (*b*) *the final velocity and the space passed through in* $t = 1, 2, 3$ *sec.*

Ans. We have $\frac{\delta}{\Delta} = 7.2$, $f = 27.7$.

Therefore, (*a*) $k = 23.06$ ft. per sec.

(*b*) For $t =$	1	2	3 sec.
$v =$	19.22	22.68	23.02 ft. per sec.
$s =$	11 38	32.8	53.94 ft.

We see that the maximum velocity is reached in about 3 sec.

In the preceding examples it is assumed that the density of the medium is unchanged, and that the acceleration of gravity is constant. Near the earth's surface both assumptions are practically true. We have also assumed that the acceleration varies as the square of the velocity. Experiments would seem to indicate that this is not strictly accurate.

The effect of resisting media upon the motion of projectiles is therefore best taken account of by means of empirical formulas based upon experiment.

We have given the preceding examples in order to call attention to the fact that the influence of the medium, even of the air, is such as to very materially modify the results of the formulas of page 93, which hold good only in vacuo and are not even approximately true except for large and heavy bodies for the first few seconds of fall. The examples of page 95 are therefore devoid of practical value except under such limitations.

For very great distances the density of the air and the acceleration of gravity are not constant, so that our present assumptions are then no longer in accord with fact.

CHAPTER III.

TRANSLATION IN A CURVED PATH

DIRECTION OF ACCELERATION CONSTANT. PARABOLIC MOTION. MOTION OF PROJECTILE IN A RESISTING MEDIUM.

Curved Path.—When a point moves in a path such that the direction of the acceleration coincides with the direction of motion and does not change, the motion is rectilinear, no matter what the law of variation of the magnitude of the acceleration may be. Such motion we have discussed in the preceding Chapters.

If the direction of the acceleration, however, does not coincide with that of the motion, then, whether it is constant in direction and magnitude or not, we have motion in a curved path.

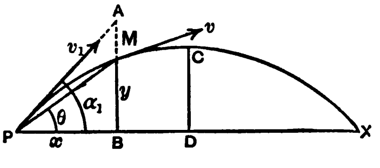
When a rigid body composed of many points moves so that every straight line through any two of its points remains parallel to itself in all positions of the body, it has a motion of translation only, and we may treat the body as if it were a point.

For motion in a curved path the differential equations of page 81 apply.

Uniform Acceleration Inclined to Direction of Motion.—If the acceleration is uniform, that is, constant in magnitude and direction, its component in any given direction is uniform, and the equations for rectilinear motion, page 93, apply to the component motion *in that direction.*

The most common case of curvilinear motion under uniform acceleration is that of a body projected with any given velocity in any given direction at the surface of the earth, neglecting the resistance of the air. In such case the acceleration due to gravity is practically uniform and equal to g ft.-per-sec. per sec.

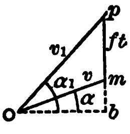
Let the initial velocity of projection v_1 of the point P make the angle $APB = \alpha_1$ with the horizontal.



Let the co-ordinates of any point M of the path or *trajectory* be $PB = x$ and $BM = y$.

Let the angle $MPB = \theta$. Let f be the uniform vertical acceleration (in the case of gravity $f = g$).

If we make Op parallel and equal to the velocity v_1 at P, and Om parallel and equal to the velocity v at any point M of the trajectory, then $pm = ft$ is the integral acceleration for the time t during which the point passes from P to M, and the straight line pb is the *hodograph* (page 52)



for motion from P to the point C, where the velocity is horizontal, and Ob is the velocity at this point.

The velocity of p in the hodograph is the acceleration in the path (page 52). Therefore the point p moves with uniform velocity f from p to b, while P moves from P to C.

We see at once that the horizontal component of the velocity v_1 is $Ob = v_x$, or

$$Ob = v_x = v_1 \cos \alpha_1. \qquad (1)$$

The horizontal distance passed over in any time t, while the point P moves from P to M, is then $PB = x$, or

$$PB = x = v_1 t \cos \alpha_1. \qquad (2)$$

The vertical component of the velocity v_1 is $bp = v_1 \sin \alpha_1$ upwards. But the acceleration f is downwards. Hence the vertical velocity at the end of the time t is $bm = bp - pm = v_y$, or

$$bm = v_y = v_1 \sin \alpha_1 - ft. \qquad (3)$$

The vertical velocity at the beginning of the time t is $v_1 \sin \alpha_1$. The *mean vertical velocity* during the time t is then

$$\frac{2v_1 \sin \alpha_1 - ft}{2} = v_1 \sin \alpha_1 - \frac{1}{2}ft.$$

The vertical distance passed through in the time t is then $BM = y$, or

$$BA - AM = BM = y = v_1 t \sin \alpha_1 - \frac{1}{2}ft^2. \qquad (4)$$

If we combine (2) and (4) by eliminating t, we have for the *equation of the trajectory*

$$y = x \tan \alpha_1 - \frac{fx^2}{2v_1^2 \cos^2 \alpha_1}. \qquad (5)$$

This is the equation of a *parabola*.

The time of reaching the highest point C is the time of describing the vertical distance DC. Call this time T_v. Since at this point the vertical velocity is zero, we have, by making $v_y = 0$ in (3),

$$T_v = \frac{v_1 \sin \alpha_1}{f}.$$

If we substitute this for t in (2) and (4) we obtain the co-ordinates of the vertex C of the parabola,

$$PD = x_0 = \frac{v_1^2 \sin \alpha_1 \cos \alpha_1}{f} = \frac{v_1^2 \sin 2\alpha_1}{2f};$$

$$DC = y_0 = \frac{v_1^2 \sin^2 \alpha_1}{2f} = \frac{x_0^2 f}{2v_1^2 \cos^2 \alpha_1}.$$

The parameter of the parabola is then $\frac{x_0^2}{2y_0} = \frac{v_1^2 \cos^2 \alpha_1}{f}$. The directrix is parallel to PD at a distance above the vertex C equal to one half the parameter or $\frac{v_1^2 \cos^2 \alpha_1}{2f}$, or at a distance of $\frac{v_1^2}{2f}$ above P. That is, *distance of the directrix above P is the height due to the velocity v_1.*

If we had taken the origin at the vertex C and let x_c and y_c be the new co-ordinates, then the horizontal velocity at C would be

$v_1\cos\alpha$, and the horizontal distance passed over in any time t would be $x_c = v_1 t\cos\alpha_1$. The mean vertical velocity would be $\frac{1}{2}ft$ and the vertical distance $y_c = \frac{1}{2}ft^2$. Combining and eliminating t, we obtain

$$x_c^2 = \frac{2v_1^2\cos^2\alpha_1}{f}y_c,$$

which is the equation of a parabola referred to its diameter CD and the tangent at the vertex C. The parameter is as before $\frac{v_1^2\cos^2\alpha_1}{f}$.

To find the velocity at any point of the trajectory.—The magnitude of the velocity at any point M is the resultant of the vertical and horizontal velocities, or, from (1) and (3),

$$v^2 = v_x^2 + v_y^2 = v_1^2 - 2v_1 ft\sin\alpha_1 + f^2t^2. \quad . \quad . \quad . \quad . \quad (6)$$

The same result is obtained at once from the hodograph from the triangle Opm.

Inserting the value of y from (4),

$$v^2 = v_1^2 - 2fy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the acceleration is due to gravity, we replace f by g, and have $\frac{v^2}{2g} = \frac{v_1^2}{2g} - y$. But we have just seen that $\frac{v_1^2}{2g}$ is the distance of the directrix above P. Therefore $\frac{v_1^2}{2g} - y$ is the distance of the directrix above any point M, and $\frac{v^2}{2g}$ is the height due to the velocity v. Hence, *the speed at any point is the same as that acquired by a body falling from the directrix to that point.*

To find the *direction* of the velocity v at any point M, the magnitude of which is given by (6) and (7), let α be the angle which it makes with the horizontal. Then we have directly from the hodograph, since angle $mOb = \alpha$,

$$v\sin\alpha = v_1\sin\alpha_1 - ft\,;$$

$$v\cos\alpha = v_1\cos\alpha_1.$$

Therefore, from (2),

$$\tan\alpha = \tan\alpha_1 - \frac{ft}{v_1\cos\alpha_1} = \tan\alpha_1 - \frac{ft^2}{x}, \quad . \quad . \quad . \quad (8)$$

or

$$\tan\alpha = \tan\alpha_1 - \frac{fx}{v_1^2\cos^2\alpha_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

To find the time of flight in a horizontal direction, and the horizontal range.—If in (4) we make $y = 0$, we have for the time T_h in which the body reaches the line PX, or the time of flight in a horizontal direction,

$$T_h = \frac{2v_1\sin\alpha_1}{f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Inserting this value of t in (2), we have for the horizontal range PX,

$$R_h = \frac{2v_1^2\sin\alpha_1\cos\alpha_1}{f} = \frac{v_1^2\sin 2\alpha_1}{f}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is twice the distance PD.

The greatest value $\sin 2\alpha_1$ can have is unity, and this occurs when $2\alpha_1 = 90°$ or $\alpha_1 = 45°$. Therefore the horizontal range, neglecting resistance of the air, is greatest for an angle of elevation of 45°, and is equal to $\frac{v_1^2}{f}$.

To find the greatest height attained, and the corresponding time.—Put the vertical velocity given by (3) equal to zero, and we have for the time of attaining the greatest height

$$T_v = \frac{v_1 \sin \alpha_1}{f}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

or just half the whole time of flight as given by (10).

Insert the value of the time given by (12) in (4), and we have for the greatest height attained, $CD = H$,

$$H = \frac{v_1^2 \sin^2 \alpha_1}{2f}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

Equations (13) and $\frac{1}{2}R_h$ given by (11) give the *co-ordinates of the vertex C.*

To find the displacement in any given direction, and the corresponding time.—Let θ be the angle which any displacement $PM = R$ makes with the horizontal, then we have $BM = y = x \tan \theta$. Substituting this value of y in (5), we have for the abscissa of the point M

$$x = \frac{2v_1^2 \cos \alpha_1 \sin (\alpha_1 - \theta)}{f \cos \theta} = \frac{v_1^2[\sin (2\alpha_1 - \theta) - \sin \theta]}{f \cos \theta}, \quad . \quad (14)$$

and therefore for the displacement or range $PM = R$,

$$R = \frac{2v_1^2 \cos \alpha_1 \sin (\alpha_1 - \theta)}{f \cos^2 \theta} = \frac{v_1^2[\sin (2\alpha_1 - \theta) - \sin \theta]}{f \cos^2 \theta}. \quad . \quad (15)$$

If in (15) we make $\theta = 0$, we have the horizontal range R_h as given by (11).

If we divide (14) by the horizontal component of the velocity, $v_1 \cos \alpha_1$, we have

$$t = \textit{time of flight} = \frac{2v_1 \sin (\alpha_1 - \theta)}{f \cos \theta}. \quad . \; . \; . \; . \quad (16)$$

This reduces to (10) for $\theta = 0$.

To find the angle of elevation which gives the greatest range in any given direction.—The range R given by (15) is a maximum when $\sin (2\alpha_1 - \theta)$ is a maximum, or when $2\alpha_1 - \theta = 90°$ or $\alpha_1 = \frac{1}{2}\left(90° + \theta\right)$. The direction of projection for the greatest range makes therefore with the vertical an angle $90 - \alpha_1 = \frac{1}{2}\left(90 - \theta\right)$, that is, it *bisects the angle between the vertical and the range.*

To find the elevation necessary to hit a given point.—To determine the direction of the velocity v_1 in order that the path may pass through a given point given by x and y, we substitute for $\frac{1}{\cos^2 \alpha_1}$ the equivalent value $1 + \tan^2 \alpha_1$ in equation (5), and obtain at once

$$\tan \alpha_1 = \frac{v_1^2}{fx} \pm \sqrt{\left(\frac{v_1^2}{fx}\right)^2 - \left(1 + \frac{2v_1^2 y}{fx^2}\right)}. \quad . \; . \; . \; . \quad (17)$$

Also from (14) we have

$$\alpha_1 = \frac{\theta}{2} + \frac{1}{2}\sin^{-1}\left(\frac{fx\cos\theta}{v_1^2} + \sin\theta\right),$$

or, since $R\cos\theta = x$,

$$\alpha_1 = \frac{\theta}{2} + \frac{1}{2}\sin^{-1}\left(\frac{fR\cos^2\theta}{v_1^2} + \sin\theta\right). \quad \ldots\ldots \quad (18)$$

We see from (17) that α_1 has two values. If α_1' is an angle such that $\sin(2\alpha_1' - \theta) = \sin(2\alpha_1 - \theta)$, then $2\alpha_1' - \theta = 180^\circ - (2\alpha_1 - \theta)$ or $\alpha_1' = 90 - (\alpha_1 - \theta)$ and either α_1' or α_1 will satisfy equation (14).

With a given acceleration and initial velocity of projection of given magnitude, there are therefore *two directions* of the initial velocity, α_1 and $90 - (\alpha_1 - \theta)$, and therefore *two paths* by which the body may attain the same point.

If in (17) we put $\left(\frac{v_1^2}{fx}\right)^2 = 1 + \frac{2v_1^2 y}{fx^2}$, we have

$$v_1^2 = f(y + \sqrt{x^2 + y^2}) \quad \text{and} \quad \tan\alpha_1 = \frac{v_1^2}{fx}.$$

Smaller values of v_1 make $\tan\alpha_1$ imaginary. Larger values of v_1 give two values for $\tan\alpha_1$. In the first case the point cannot be attained. In the second case it would be attained either in the rise or fall of the projectile.

To find the envelope of all the trajectories corresponding to different values of α_1 for a given initial speed v_1. — Equation (5) gives the equation of the trajectory corresponding to the angle of elevation α_1. If we substitute $1 + \tan^2\alpha_1$ for $\frac{1}{\cos^2\alpha_1}$, equation (5) becomes

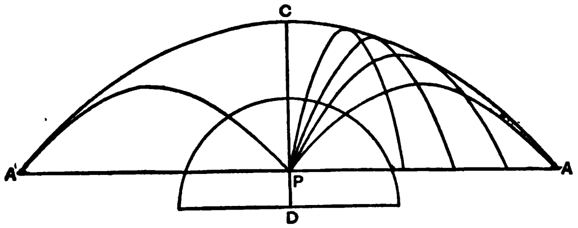
$$y = x\tan\alpha_1 - \frac{fx^2(1 + \tan^2\alpha_1)}{2v_1^2}, \quad \ldots\ldots \quad (19)$$

where x and y are the co-ordinates of any point of the path.

For another angle of elevation α_1', and the same initial speed v_1, we have

$$y_1 = x_1\tan\alpha_1' - \frac{fx_1^2(1+\tan^2\alpha_1')}{2v_1^2},$$

where x_1 and y_1 are the co-ordinates of any point of the new trajectory.



If we make $x = x_1$ and $y = y_1$, we have for the point of intersection of the two trajectories, by equating these two equations,

$$\frac{fx}{2v_1^2}(\tan\alpha_1 + \tan\alpha_1') = 1.$$

If the angles α_1' and α_1 approach equality, this expression approaches the limit

$$\frac{fx}{v_1^2}\tan\alpha_1 = 1, \text{ or } \tan\alpha_1 = \frac{v_1^2}{fx}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Equation (20) gives then the value of $\tan\alpha_1$ when the two trajectories starting from the same point P with the same speed v_1 have angles of elevation at P whose difference is indefinitely small.

Substituting this value of $\tan\alpha_1$ in (19) we obtain

$$y = \frac{v_1^2}{2f} - \frac{fx^2}{2v_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Equation (21) is then the equation of a curve which passes through all the points in which every two trajectories starting from the same point P at angles of elevation whose difference is indefinitely small cut each other. It is therefore the equation of the *envelope* or curve which touches all the trajectories or parabolas described from the same point P with the same initial speed v_1.

Equation (21) is the equation of a parabola ACA', whose axis PC is vertical, whose focus is the point P of projection, and whose vertex C is in the common direction of the trajectories.

With the given initial speed v_1, the projectile can reach any point within this envelope by two angles of elevation and two trajectories, as proved page 121. It can reach any point in the envelope by only one elevation and path. It cannot reach with any elevation and with the given velocity v_1 any point outside this envelope.

The point, therefore, where this envelope cuts the plane of any given range gives the *maximum range in that direction* for any given v_1.

Thus the maximum range on a horizontal plane is found from (21), by making $y = 0$, to be $\frac{v_1^2}{f}$. The same result is given by (15) when we make $\alpha_1 = 45°$ and $\theta = 0$.

Questions of maximum range may thus be readily solved by the equation for the envelope.

From (2) we have $\cos\alpha = \frac{x}{v_1 t}$, and from (4) $\sin\alpha_1 = \frac{y + \frac{1}{2}ft^2}{v_1 t}$.

Since $\cos^2\alpha_1 + \sin^2\alpha_1 = 1$, we have

$$x^2 + \left(y + \frac{1}{2}ft^2\right)^2 = v_1^2t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This is the equation of a *circle* whose radius is $v_1 t$ and whose centre is situated vertically below P at a distance $PD = \frac{1}{2}ft^2$.

The circumference of this circle is reached in the same time by a point starting from P with the velocity v_1 in any direction.

[Application of the Calculus.] The same results are obtained by the application of the differential equations of motion, page 81.

Thus in the present case we have for the horizontal component of the acceleration, since f is vertical,

$$\frac{d^2x}{dt^2} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and for the vertical component

$$\frac{d^2y}{dt^2} = -f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Integrating (*a*), since for $t = 0$, $\frac{dx}{dt} = v_1 \cos \alpha_1$, we have

$$v_x = \frac{dx}{dt} = v_1 \cos \alpha_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating again, since for $t = 0$, $x = 0$, we have

$$x = v_1 t \cos \alpha_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating (*b*), we have, since when $t = 0$, $\frac{dy}{dt} = v_1 \sin \alpha_1$,

$$v_y = \frac{dy}{dt} = v_1 \sin \alpha_1 - ft. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating again, since for $t = 0$, $y = 0$, we have

$$y = v_1 t \sin \alpha_1 - \frac{1}{2} f t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Combining (2) and (4) by eliminating *t*, we have for the equation of the trajectory

$$y = x \tan \alpha - \frac{f x^2}{2 v_1^2 \cos^2 \alpha_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We have also

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2,$$

or, from (1) and (2),

$$v^2 = v_1^2 - 2ftv_1 \sin \alpha_1 + f^2 t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Inserting the value of *y* from (4),

$$v^2 = v_1^2 - 2fy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we differentiate (5), we have, for the tangent of the angle which the velocity at any point makes with the horizontal,

$$\tan \alpha = \frac{dy}{dx} = \tan \alpha_1 - \frac{fx}{v_1^2 \cos^2 \alpha_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

These are the same equations as already given, and from them all the others are deduced.

EXAMPLES.

$g = 32.16$ ft.-per-sec. per sec. Resistance of air neglected.

(1) *Show that for parabolic motion the hodograph is a straight line.*

(2) *The sights of a gun are set so that the ball may strike a given object. Show that when the sights are directed to any other object in the same vertical line, the ball will also strike it.*

(3) *Two bodies projected from the same point in directions making angles β, β' with the vertical pass through the same point in*

the horizontal plane through the point of projection. If t and t' are the times of flight, show that

$$\frac{\sin(\beta - \beta')}{\sin(\beta + \beta')} = \frac{t'^2 - t^2}{t'^2 + t^2}.$$

(4) ***With what velocity must a projectile be fired at an elevation of 30° so as to strike an object at the distance of 2500 ft. on an ascent of 1 in 40?***

Ans. 311.5 ft. per sec.

(5) ***Find the direction and magnitude of the velocity of projection in order that a projectile may reach its maximum height at a point whose horizontal and vertical distances from the starting-point are b and h respectively.***

Ans. $\tan \alpha_1 = \frac{2h}{b}$, $v_1 = \sqrt{\frac{(4h^2 + b^2)g}{2h}}$.

(6) *A gun is fired horizontally at a height of 144.72 ft. above the surface of a lake and the initial speed of the ball is 1000 ft. per sec. Find (a) after what time, and (b) at what horizontal distance, the ball strikes the lake, neglecting resistance of the air.*

Ans. (*a*) 3 sec.; (*b*) 3000 ft.

(7) *In the parabola described by a projectile, its speed at any point is that which it would have had had it fallen to that point from the directrix.*

(8) *A particle projected at a given elevation with an initial speed v_1 reaches the top of a tower h ft. high and 2h ft. from the point of projection in t seconds. Find (a) the initial speed of another particle which, being projected at the same elevation from a point distant 4h ft. from the tower, will also reach its summit, and (b) the time it will require.*

Ans. (*a*) $\frac{v_1 t \sqrt{2g}}{\sqrt{h + gt^2}}$; (*b*) $\frac{\sqrt{2(h + gt^2)}}{\sqrt{g}}$.

(9) *A ball is projected with a velocity of 100 ft. per sec. inclined 75° to the horizon. Find (a) the range on a horizontal plane; (b) the range on a plane inclined 30° to the horizon; (c) what other directions of the initial velocity would give the same ranges.*

Ans. (*a*) 155.5 ft.; (*b*) 207.3 $(\sqrt{3} - 1)$; (*c*) 15° and 45°.

(10) *Show that with a given initial speed the greatest range on a horizontal plane is just half as great as the greatest range down an incline of 30°.*

(11) *Show that if two particles meet which have been projected with the same initial speed, in the same vertical plane, at the same instant, from two given points, the sum of their elevations must be constant.*

(12) *On a small planet a stone projected with a speed of 50 ft. per sec. is found to have a maximum range on a horizontal plane of 400 ft. Find the acceleration of falling bodies at the surface of that planet.*

Ans. 6 25 ft.-per-sec. per sec.

(13) *Two stones thrown at the same instant from points 20 yards apart, with initial velocities inclined 60° and 30°, respectively, to the horizon, strike a flag-pole at the same point at the same instant.*

Show that the initial speeds are as $1 : \sqrt{3}$; *and that the distance of the pole from the nearer point of projection is* 10 *yards.*

(14) *At what elevation must a body be projected with a speed of* 310.8 *ft. per sec. that it may hit a balloon* 500 *ft. from the earth's surface and at a distance of* 1000 *ft. from the point of projection.*

Ans. 39° 17′, or 80° 43′.

(15) *A body is projected with an initial velocity of* 30 *ft. per sec. inclined* 60° *to the horizon. Find the velocity after* 20 *sec.*

Ans. 617.3 ft. per sec. inclined 148° 36′.6 to the direction of the initial velocity.

(16) *If from a point A bodies are projected at the same moment and in the same vertical plane at different angles of elevation, with the same initial speed,* v_1, *the locus of all the positions occupied at the end of a given time* t *is a circle whose radius is* v_1t *and whose centre is situated vertically below A at a distance* $\frac{1}{2}gt^2$.

(17) *A jet of water rises with a velocity of* 20 *ft. per sec. at an angle of elevation of* 66°. *Find* (*a*) *the height due to the velocity;* (*b*) *the greatest height of the jet;* (*c*) *the horizontal range;* (*d*) *the time of reaching the horizontal plane;* (*e*) *the height corresponding to the horizontal distance* 3 *ft.*

Ans. (*a*) 6.2 ft.; (*b*) 5.17 ft.; (*c*) 9.24 ft.; (*d*) 1.14 sec.; (*e*) 4.52 ft.

(18) *A jet of water discharged horizontally at a height above a horizontal plane of* $1\frac{1}{4}$ *ft. has a range on the horizontal plane of* $5\frac{1}{4}$ *ft. Find the velocity of projection.*

Ans. 15.92 ft. per sec.

(19) *Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.*

(20) *Show that the envelope of all the parabolas which correspond to a given velocity of projection is equal to the trajectory for which the direction of projection is horizontal.*

(21) *A particle is projected over a triangle from one end of the horizontal base and, grazing the vertex, falls upon the other end of the base. If* β *and* γ *are the base angles and* α *the angle of projection, show that* $\tan \alpha = \tan \beta + \tan \gamma$.

(22) *For the greatest range on an inclined plane through the point of projection the direction of motion on leaving is at right angles to that on reaching the plane.*

(23) *A particle is projected horizontally with a speed of* 32.16 *ft. per sec. from a point* 128.64 *feet from the ground. Find the direction of its motion when it has fallen half way to the ground.*

Ans. Inclination to the horizontal $= \tan^{-1} 2$.

(24) *The greatest range on a horizontal plane of a projectile with a given initial speed being* 500 *meters, show that the greatest range on a plane inclined* 60° *to the horizontal is* $2 - \sqrt{3}$ *kilometers.*

(25) *A stone is let fall in a railway-carriage travelling at the rate of* 30 *miles per hour. Find its displacement relative to the road at the end of* 0.1 *sec.*

Ans. 4.4029 feet, inclined 2° 5′.5 to the horizon.

(26) *The velocities of a projectile at any two points of its path being given, find the difference of the altitudes above a horizontal plane.*

Ans. $\frac{V_1^2 - V_2^2}{2g}$, where V_1 and V_2 are the magnitude of the given velocities.

(27) *A given inclined plane passes through the point of projection of a projectile which eventually strikes the plane at right angles. Find the range of the projectile on the inclined plane, the velocity of projection being given.*

Ans. If θ is the inclination of the plane and v_1 the velocity of projection, the required range is $\frac{2v_1^2}{g} \frac{\sin\theta}{1 + 3\sin^2\theta}$.

(28) *A particle begins to slide from rest down an inclined plane AB. At the same instant another particle is projected from A. Find the condition that the particles may meet, and ascertain when and where this occurs.*

Ans. The second particle must be projected at right angles to the plane. If θ is the inclination of the plane and v_1 the velocity of projection, the time before meeting is $t = \frac{2v_1}{g\cos\theta}$, and the distance of the point of meeting from A will be $\frac{2v_1 \sin\theta}{g\cos^2\theta}$.

(29) *Prove that the components of the velocities at the extremities of any chord of the path of a projectile, at right angles to the chord, are equal.*

(30) *Swift of foot was Hiawatha;*
He could shoot an arrow from him,
And run forward with such fleetness,
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward,
Shoot them with such strength and swiftness,
That the tenth had left the bow-string
Ere the first to earth had fallen.

Supposing Hiawatha to shoot an arrow every second, and, when not shooting vertically, to have aimed so that the flight of the arrow might have the longest range, find his speed.

Ans. About 99 miles an hour.

(31) *If any number of bodies are projected from the same point in different directions with the same initial speed* v_1, *show that the foci of the parabolas they will describe will lie on the surface of a sphere whose radius is* $\frac{v_1^2}{2g}$.

(32) *The elevation of a projectile is that for maximum horizontal range. Find the time of reaching a point whose horizontal and vertical distances from the point of projection are h and k respectively.*

Ans. $t = \sqrt{\frac{2(h-k)}{g}}$.

(33) *If AB is the range of a projectile on a horizontal plane, and t the time from A to any point P of the trajectory, and t′ the time from P to B, show that the height of P above AB is* $\frac{1}{2}gtt'$.

(34) *A projectile is fired from an elevation of* 1.8 *meters above a horizontal plane with a horizontal velocity of* 10 *meters per sec. How long before it strikes the plane and what is the range?* ($g = 9.81$ *meters-per-sec. per sec.*)

Ans. 0.6 sec.; 6 meters.

(35) *A projectile is fired at an angle of* 30° *at a target distant* 1200 *meters in a horizontal direction.* ($g = 9.81$ *meters-per-sec. per sec.*)

(*a*) *Find the initial velocity.*
(*b*) *The time of striking.*
(*c*) *The highest point of the trajectory.*
(*d*) *The velocity of striking.*

Ans. (*a*) 116.6 meters per sec.; (*b*) about 12 seconds; (*c*) 173.21 meters; (*d*) same as the initial velocity.

(36) *A projectile is fired with an initial velocity of* 150 *meters per sec. from a point* 100 *meters below a target which is distant horizontally* 1525 *meters.* ($g = 9.81$ *meters-per-sec. per sec.*)

(*a*) *Find the angle of elevation.*
(*b*) *The velocity of striking.*
(*c*) *The time of flight.*

Ans. (*a*) 68° 28′ 50″ for bomb, 25° 16′ for ball; (*b*) 143.3 meters per sec.; (*c*) 27.715 sec. for bomb, 11.24 sec. for ball.

(37) *A projectile is fired at an elevation of* 45°, *and strikes a target at a horizontal distance of* 800 *meters and* 70 *meters lower.* ($g = 9.81$ *meters-per-sec. per sec.*)

(*a*) *Find the initial velocity.*
(*b*) *The final velocity.*
(*c*) *The angle of striking.*

Ans. (*a*) 84.95 meters per sec.; (*b*) 92.6 meters per sec.; (*c*) 49° 33′.

[**Motion of a Projectile in a Resisting Medium.**]—In the preceding examples the motion is assumed to be *in vacuo*. It should be borne in mind that the resistance of the air completely changes the results of the theoretic formulas.

The motion of a projectile, taking into account the resistance of the air, is best given by empirical formulas based upon experiment.

If, however, we assume that the magnitude of the acceleration decreases directly with the square of the velocity, we may deduce by means of our general differential equations, page 81, Chap. VIII, the equation of the trajectory for very small angles of elevation. The same method which we shall use holds good for any other assumption as to the law of acceleration.

The assumption is not strictly accurate, but will serve to illustrate the method of deduction.

Let f be the constant acceleration due to gravity, the value of which is given page 111, and let c be the coefficient of resistance, so that c has the value given page 113.

Thus, since the velocity at any point is $v = \frac{ds}{dt}$, we have the retardation $a = -c\left(\frac{ds}{dt}\right)^2$ by assumption. We have then from equation (6), page 81, since $\cos\alpha = \frac{dx}{ds}$, for the horizontal component of the tangential acceleration

$$\frac{d^2x}{dt^2} = -c\left(\frac{ds}{dt}\right)^2\frac{dx}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, since $\sin\alpha = \frac{dy}{ds}$, for the vertical component of the tangential acceleration

$$\frac{d^2y}{dt^2} = -f - c\left(\frac{ds}{dt}\right)^2\frac{dy}{ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by $\frac{dx}{dt}$, we have

$$\frac{d^2x}{dt} \times \frac{dt^2}{dx} = -c\,ds.$$

Integrating this, we have

$$\operatorname{logn} \frac{dx}{dt} = -cs + C, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

where C is a constant of integration. Let $v_1 \cos \alpha_1$ be the component of the initial velocity v_1, parallel to x, α_1 being the angle of elevation at the point of projection; then when $s = 0$, $\frac{dx}{dt} = v_1 \cos \alpha_1$, and $C = \operatorname{logn} v_1 \cos \alpha$. Therefore, from (3),

$$\frac{dx}{dt} = e^{-cs} v_1 \cos \alpha_1, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (4)$$

where e is the base of the Naperian system of logarithms, 2.718282. Hence

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = e^{-cs} v_1 \cos \alpha_1 \frac{dy}{dx}. \quad . \ . \ . \ . \ . \ . \quad (5)$$

If we multiply (2) by dx, and (1) by dy, and subtract, we have

$$\frac{d^2y\,dx - d^2x\,dy}{dt^2} = -f\,dx. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (6)$$

Inserting the value of dt^2 from (4), we have

$$\frac{d^2y\,dx - d^2x\,dy}{dx^2} = -\frac{fe^{2cs}}{v_1^2 \cos^2 \alpha_1} dx. \quad . \ . \ . \ . \ . \ . \ . \quad (7)$$

The first member of (7) is equal to $d\frac{dy}{dx}$. We have also

$$dx^2 + dy^2 = ds^2, \quad \text{or} \quad \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = \frac{ds}{dx}, \quad \text{or} \quad dx = \frac{ds}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}}.$$

Substituting these values in (7), we have

$$\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} d\frac{dy}{dx} = -\frac{fe^{2cs}}{v_1^2 \cos^2 \alpha_1} ds. \quad . \ . \ . \ . \ . \ . \quad (8)$$

If we assume the angle of projection α_1 as very small, so that the trajectory is very flat, we have approximately in such case

$$ds = dx, \quad \text{and} \quad s = x, \quad \text{and} \quad \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = 1.$$

Therefore (8) becomes

$$d\frac{dy}{dx} = -\frac{fe^{2cx}}{v_1^2 \cos^2 \alpha_1} dx. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

Integrating (9), since when $x = 0$, $\frac{dy}{dx} = \tan \alpha_1$, we have

$$\frac{dy}{dx} = \tan \alpha_1 - \frac{f}{2cv_1^2 \cos^2 \alpha_1}\left(e^{2cx} - 1\right). \quad . \ . \ . \ . \ . \ . \quad (10)$$

Integrating (10), we have, since for $x = 0$, $y = 0$,

$$y = x \tan \alpha_1 + \frac{fx}{2cv_1^2 \cos^2 \alpha_1} - \frac{f}{4c^2v_1^2 \cos^2 \alpha_1}\left(e^{2cx} - 1\right). \quad . \ . \ . \quad (11)$$

Equation (11) is the approximate equation of the path.

If we expand the last term in a series, we have

$$y = x \tan \alpha_1 - \frac{fx^2}{2v_1^2 \cos^2 \alpha_1} - \frac{fcx^2}{3v_1^2 \cos^2 \alpha_1} - \dots \quad . \quad . \quad . \quad (12)$$

If the terms containing c are omitted, and $f = g$, equation (12) is that of a parabola, which is the path of the projectile in vacuo.

The ordinate of the actual curve is therefore less at any distance x than that of the parabola for the same distance.

From equation (4) we have

$$\frac{dx}{dt} = e^{-cx} v_1 \cos \alpha_1 ;$$

and integrating, since for $x = 0$, $t = 0$, we have

$$cv_1 \cos \alpha_1 t = e^{cx} - 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which gives the time in terms of the abscissa.

From (13) we have

$$x = \frac{1}{c} \operatorname{logn} (cv_1 \cos \alpha_1 t + 1),$$

or, in common logarithms,

$$x = \frac{2.302585}{c} \log (cv_1 \cos \alpha_1 t + 1), \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which gives the abscissa x in terms of the time.

"The general problem of the path of a projectile in a uniform resisting medium, where the resistance varies as the square of the velocity, was proposed by Keill as a trial of skill to John Bernoulli, by whom the challenge was received Feb. 1718. Keill, trusting to the complexity of the analysis, which had probably deterred Newton from attempting any regular solution of the problem in the second book of the *Principia*, was in hopes that the exertions of Bernoulli would prove unsuccessful. Bernoulli, however, having expeditiously effected a solution, not only of Keill's problem, but likewise of the more general one where the resistance varies as the *n*th power of the velocity, expressed a determination not to publish his investigation until he had received intimation that his antagonist had himself been able to solve his own problem. He gave Keill till the following September to exercise his talents, declaring that if he received by that time no satisfactory communication, he should feel himself entitled to question the ability of his adversary. At the request of a common friend, Bernoulli consented to extend the interval to the first of November. It turned out, however, that Keill was unable to obtain a solution. At length Nicholas Bernoulli, Professor of Mathematics at Padua, communicated to John Bernoulli a solution of Keill's problem, which the author afterwards extended to the more general one. Finally, on the 17th of November, information was received by John Bernoulli from Brook Taylor, to the effect that he had obtained a solution. John Bernoulli published his own analysis, together with that of his nephew Nicholas, in the *Acta Erudit.* Lips. 1719 mai., p. 216." (Walton, Problems in Theoretical Mechanics.)

CHAPTER IV.

CURVILINEAR MOTION OF TRANSLATION—CENTRAL ACCELERATION. HARMONIC AND PLANETARY MOTION.

Central Acceleration.—If the acceleration of a moving point is always directed towards a fixed point or centre of acceleration, the acceleration is said to be **central**.

The velocity of the moving point at any instant is the resultant of the velocity at the preceding instant and of the integral acceleration during the intervening time.

But if the acceleration is always directed towards the centre, or fixed point, its moment with reference to that point is zero. Since the moment of the resultant of any two components about any point is equal to the sum of the moments of the components, and since in this case the moment of one of the components, viz., the acceleration, is zero, it follows that *the moment of the velocity about the centre, in the case of central acceleration, is constant.*

Conversely, if the moment of the velocity of a moving point about any fixed point is constant, *the acceleration must always be directed towards that point.*

If r is the distance of the moving point from the fixed point, p the lever-arm, and ω is the angular speed at any instant, we have for the moment of the velocity $pv = r^2\omega = c$, equal to twice the areal velocity of the radius vector (page 76).

Therefore, in all cases of central acceleration $\frac{1}{2}r^2\omega$ is constant, or *the area described by the radius vector in a unit of time is constant.*

It follows also that in all cases of central acceleration $\omega = \frac{c}{r^2}$, or *the angular speed is inversely as the square of the radius vector.*

Cases of Central Acceleration.—The two most important cases of central acceleration are those of *harmonic motion*, where the central acceleration is directly proportional to the distance from the centre, and *planetary motion*, where it is inversely as the square of the distance.

When the velocity is in the same straight line as the central acceleration we have in both these cases rectilinear motion. The first is simple rectilinear harmonic motion, the second is rectilinear planetary motion or that of a body at great distances from the earth. Both these cases have been considered in Chaps. I and II.

When the velocity is not in the same straight line with the central acceleration we have *compound harmonic motion* and planetary motion in general. The first is of great importance in the study of sound, light, heat, etc., as well as in ordinary kinetics. The second is the motion of planets about the sun and of satellites about their primaries.

Cases of Harmonic Motion.—We have defined simple harmonic motion, page 103, as the motion of a point moving in any path in such a manner that the *tangential* component of the acceleration, a, is directly proportional to the distance, measured along the path, from a fixed point in the path.

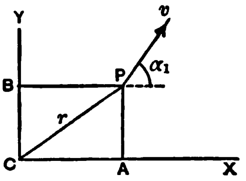
Such motion may be rectilinear or curvilinear. In the first case it is simple rectilinear, in the second simple curvilinear.

If the whole acceleration itself, or f, is central, that is, always directed towards a fixed point not in the path, and is always proportional to the distance from this fixed point, the motion is *central harmonic*, or *compound harmonic*, so called, because it is the resultant of two simple rectilinear motions, as will be proved in the next article.

Simple rectilinear harmonic motion is also *central*, because the fixed point is in the path.

Any Central Harmonic Motion may be Resolved into Two Simple Rectilinear Harmonic Motions at Right Angles.—Let C be the centre of acceleration, and P the position of the moving point at any instant. Let the velocity v of P make an angle α with the axis of X, and let the motion of P be harmonic so that the acceleration of P is $\frac{a'}{r'}r$, where a' is the acceleration at a known distance r', and r is the distance CP.

The velocity v may be resolved into $v \cos \alpha$ and $v \sin \alpha$ in the directions CX and CY, and the acceleration may be resolved into $\frac{a'}{r'}r \cos PCA$ or $\frac{a'}{r'}\overline{CA},$ and $\frac{a'}{r'}r \cos PCB$ or $\frac{a'}{r'}\overline{CB},$ in the same directions.



The component accelerations are therefore directly as the distances CA and CB, and the component velocities are in the directions of CA and CB. The central harmonic motion of P, whatever the direction of the velocity v, is therefore the resultant of two simple harmonic motions in the lines CA and CB at right angles.

If then any central harmonic motion is resolved into two components at right angles, the component motions are rectilinear harmonic.

Conversely, *the resultant of two rectilinear harmonic motions at right angles is a central harmonic motion.* Central harmonic motion is therefore called *compound harmonic* motion.

Composition of the Simple Rectilinear Harmonic Motions in Different Lines.—Let the point M move in a circle AMA' of radius $r = CA = CM$ with a constant angular velocity ω. Then the motion of the projection P in the line AA' is simple rectilinear harmonic (page 103).

Let the point M_1 move in the circle CBM_1 of radius $r_1 = CB = CM_1$, with constant angular velocity ω_1. Then the motion of the projection P_1 in the line CB is simple rectilinear harmonic. Let the angle BCA between the planes of the circles be α.

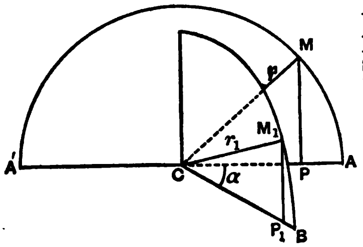


FIG. 1.

Let the time count from the instant when M_1 is at B, so that the *epoch* of P_1 is zero (page 105). At this instant let the epoch of P be ϵ. Then ϵ is the difference of epoch, or, in angular measure, the angle of M above or below A at the beginning of the time. In any time t, M_1 will have moved from B through the angle $\omega_1 t$ measured from CB, and M through the angle $\omega t \pm \epsilon$ measured from CA.

By the preceding Article we can resolve the harmonic motion of P_1 into a simple rectilinear harmonic motion at right angles to CA, and another along CA.

The displacement of P_1 from C for any time t is $r_1 \cos(\omega_1 t)$, and this displacement may be resolved into $r_1 \cos\alpha \cos(\omega_1 t)$ along CA, and $r_1 \sin\alpha \cos(\omega_1 t)$ perpendicular to CA. The displacement of P from C in the same time t is $r \cos(\omega t \pm \epsilon)$.

If a point undergoes these displacements simultaneously, its resultant displacement along CA will be

$$x = r\cos(\omega t \pm \epsilon) + r_1 \cos\alpha \cos(\omega_1 t), \quad . \quad . \quad . \quad . \quad (1)$$

and perpendicular to CA

$$y = r_1 \sin\alpha \cos(\omega_1 t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation of the curve in which the point moves, referred to rectangular co-ordinates with C for the origin, will then be obtained by combining (1) and (2) so as to eliminate t. Such combination (page 131) gives always a central or compound harmonic motion about C, the radius vector from C passing over equal areas in equal times (page 130).

Equations (1) and (2) enable us then to find the curve resulting from the combination of any two simple rectilinear harmonic motions inclined at any angle α.

If the component motions are at right angles, $\alpha = 90°$. If the amplitudes are equal, $r = r_1$. If the periods are equal, $\omega = \omega_1$, the difference of epoch is constant, and, since the epoch equals the product of the phase at zero of time by 2π radians (page 105), when the periods are equal the difference of phase is constant. When, then, the periods are equal and $\epsilon = 0$, or the epochs are equal, the phases are also equal at any instant. (For definitions of amplitude, period, epoch and phase, see page 105.)

Two Component Simple Rectilinear Harmonic Motions in Different Lines with the Same Period.—In this case $\omega = \omega_1$ and ϵ is constant, or the difference of epochs is constant and difference of phase at any instant is constant.

We have then, from (1) and (2),

$$x = r\cos(\omega t + \epsilon) + r_1 \cos\alpha \cos(\omega t), \; y = r \sin\alpha \cos(\omega t).$$

Combining these two equations by eliminating ωt, we have

$$(r_1^2 \sin^2\alpha)x^2 - 2r_1 \sin\alpha(r\cos\epsilon + r_1\cos\alpha)xy + \\ (r^2 + 2rr_1\cos\epsilon\cos\alpha + r_1^2\cos^2\alpha)y^2 = r^2r_1^2\sin^2\alpha\sin^2\epsilon. \quad (3)$$

This is the equation of an ellipse referred to its centre and rectangular axes.

Hence if a point has two component simple rectilinear harmonic motions in any directions, of any amplitudes, and any difference of epoch, *if the periods of the two components are the same*, the resultant motion of the point will be central harmonic in an ellipse, the centre of acceleration at the centre of the ellipse. The areal velocity of the radius vector about the centre is constant (page 130).

Such motion is called *elliptic harmonic motion*. Elliptic harmonic motion, then, is compound harmonic motion when the periods of the components are the same.

Equation (3) gives all cases of compound harmonic motion for equal periods of the components.

It will be instructive to derive from it special cases.

(*a*) **Two Component Simple Rectilinear Motions in Different Lines with the Same Period and Phase.**—In this case we make in (3) $\epsilon = 0$, and therefore the phases are equal, and we have at once

$$x = \frac{r + r_1 \cos \alpha}{r_1 \sin \alpha} y.$$

This is the equation of a straight line passing through the centre *C*. The resultant motion is therefore central harmonic in a straight line, or simple rectilinear harmonic.

If *CA* and *CB* are the amplitudes r and r_1 inclined at the angle α, the resultant motion has the amplitude *CR*, in direction and magnitude the diagonal of the parallelogram whose adjacent sides are r and r_1, inclined at the angle α.

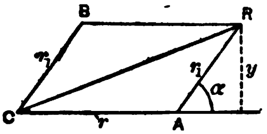


FIG. 2.

Conversely, a simple rectilinear harmonic motion whose amplitude is *CR* may be resolved, by completing the parallelogram, into two others in any two directions, *of the same period*, *epoch and phase*.

If $\alpha = 90°$, we have $y = \frac{r_1}{r}x$. Therefore the projection of a simple rectilinear harmonic motion on any straight line is also a simple rectilinear harmonic motion of the same period, epoch and phase.

If the component motions are more than two, they may be compounded two and two, and therefore *any number* of component simple rectilinear harmonic motions in any directions, of the same period, epoch and phase, give a single resultant rectilinear harmonic motion of determinate direction and amplitude, which may be resolved into two components in any two directions, of the same period, epoch and phase.

(*b*) **Two Component Simple Rectilinear Motions in the Same Line with the Same Period and Different Epochs and Phases.**—In this case we make in (3) $\alpha = 0$, and obtain at once

$$(r^2 + 2rr_1 \cos \epsilon + r_1^2)y^2 = 0.$$

But since for $\alpha = 0$, $y = 0$, (see Fig. 1,) we have

$$r^2 + 2rr_1 \cos \epsilon + r_1^2 = \text{constant}.$$

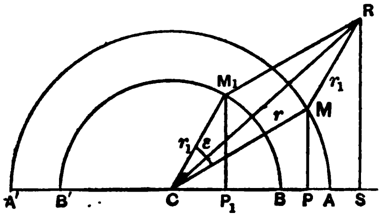


FIG. 3.

In Fig. 3 the points P and P_1 move in the line AA' with simple harmonic motion and the diagonal $CR = \sqrt{r^2 + 2rr_1 \cos \epsilon + r_1^2}$, where ϵ is the constant difference of epoch and phase.

Since ϵ is constant and *CR* is constant, its inclination to *CM* or CM_1 is constant. At any instant the resultant displacement is $CP_1 + CP = CS$, and the motion of *S* is therefore the resultant motion and is simple rectilinear harmonic, with the amplitude *CR*, the diagonal of the parallelogram on r and r_1. The epoch and phase are intermediate between the epochs and phases of the components.

If the epochs and phases are the same, $\epsilon = 0$ and the amplitude of the resultant motion is $r + r_1$, or the sum of those of the components. If the difference of epoch or phase is $\epsilon = \pi$ radians, the amplitude is $r - r_1$ or the difference of those of the components.

By taking CM_1 and CM of proper lengths we can make MCP and M_1CM what we please without changing CR. *Therefore any simple rectilinear harmonic motion may be resolved into two others in the same line, with any required difference of phase and one of them having any desired epoch, the periods being the same.*

Three or more component simple rectilinear harmonic motions in the same line and with the same period may be compounded two and two, and the resultant will be rectilinear harmonic with the same period.

If the periods are different, the angle $M_1CM = \epsilon$ will vary and CR will vary. When $\epsilon = 0$, CR will have its maximum value $r + r_1$. When the difference of epoch ϵ is π radians, CR has its minimum value $r - r_1$. The angular velocity of CR is also variable. The direction of CR will oscillate back and forth about CM, the maximum inclination being $\sin^{-1} \frac{r_1}{r}$. The resultant motion is therefore not simple rectilinear harmonic, but a more complex motion. It is, as it were, simple harmonic with periodically increasing and decreasing amplitude, and periodical acceleration and retardation of phase, or epoch.

(*c*) **Two Component Simple Rectilinear Harmonic Motions at Right Angles with the Same Period and Different Phases or Epochs.**—The general equation for this case is given by (3). If the directions are at right angles, we have $\alpha = 90°$. Suppose in addition the amplitudes equal, so that $r = r_1$, and the difference of epoch $\epsilon = 90°$. We have then, from (3),

$$x^2 + y^2 = r^2.$$

Since the motion is central harmonic, according to page 130 the areal velocity of the radius vector is constant; and since the radius is constant, the speed in the circle is constant. We have already seen, page 103, that the projection of the motion of a point moving with uniform speed in a circle, upon a diameter, gives rectilinear harmonic motion. The projection upon two diameters at right angles gives then two component rectilinear harmonic motions of the same period, with a difference of epoch of 90°, or of phase of $\frac{1}{4}$, since, when one component has its greatest displacement, the other is at the centre with displacement zero.

It follows also that two component simple rectilinear harmonic motions at right angles, with the same period and equal amplitudes, differing in epoch by 90° or in phase by one quarter of a period, will give, as a resultant, uniform motion in a circle whose radius is the common amplitude of the components.

If the amplitudes are not equal, but α and ϵ still 90°, and periods the same, we have, from (3),

$$r_1^2x^2 + r^2y^2 = r^2r_1^2,$$

which is the equation of an ellipse referred to its centre and axes.

The resultant motion is therefore central harmonic in an ellipse, whose semi-diameters are r and r_1, the centre at the centre of the ellipse.

The same result is evidently obtained by projecting the circle in the preceding case upon a plane, so as to obtain the required amplitude r_1, r remaining unchanged.

(*d*) **Three or More Component Simple Rectilinear Harmonic Motions in Different Lines with the Same Period but Different Phases or Epochs.**—We have seen from equation (3) that the resultant of two simple rectilinear component harmonic motions in any two directions, of the same period and different epoch or phase, is elliptic harmonic motion.

We have also seen from (*a*) that any simple rectilinear harmonic motion may be resolved into two others of the same period and phase or epoch in any two given directions. Any number of given simple rectilinear harmonic motions, then, of the same period and different phases or epochs may each be resolved into its own pair in any two given directions. These pairs constitute a number of simple rectilinear harmonic motions in two given lines, all of the same period and different phases or epochs.

According to (*b*), all in one line may be compounded into one resultant, and all in the other line into another resultant, these two resultants having the same period and different phases or epochs. The resultant of these two is, according to equation (3), elliptic harmonic motion.

Hence the resultant of *any number* of component simple rectilinear harmonic motions *of the same period*, whatever their amplitudes, directions, phases or epochs, is elliptic harmonic motion, the centre of the ellipse being then centre of acceleration, and the radius vector describing equal areas in equal times.

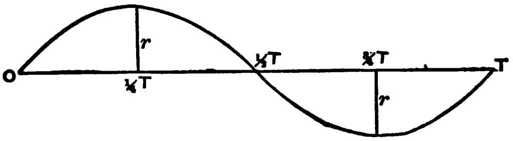
In special cases this becomes, as we have seen, uniform circular motion or simple rectilinear harmonic motion.

Since the above holds whatever the inclination of the two resultants, elliptic harmonic motion may be considered as the resultant of two component simple harmonic motions of the same period and different epochs or phases at right angles.

Graphic Representation.—We may exhibit graphically simple or compound rectilinear harmonic motion by a curve in which the abscissas represent intervals of time, and the ordinates the corresponding distance of the moving point from its mean position.

In the case of a single harmonic motion we have (page 132) $x = r\cos(\omega t \pm \epsilon)$. If the distance x is to be zero when $t = 0$, we must have the epoch $\epsilon = \frac{\pi}{2}$ radians, or one fourth of the periodic time. This gives $x = r\sin\omega t$.

Since $\omega = \frac{2\pi}{T}$, where T is the periodic time, we have for $t = 0$, $x = 0$; for $t = \frac{1}{4}T$, $x = r$; for $t = \frac{1}{2}T$, $x = 0$; for $t = \frac{3}{4}T$, $x = -r$; for $t = T$, $x = 0$.



The curve is the curve of sines, or the curve which would be described by the point P (page 103) if, while M maintained its uniform circular motion, the circle itself were to move with uniform speed in a direction perpendicular to CA.

It is the simplest possible form assumed by a vibrating string, when it is assumed that at each instant the motion of every particle of the string is simple harmonic.

If the rectilinear harmonic motion is compound, we have (page 132) in general

$$x = r\cos(\omega t \pm \epsilon) + r_1\cos(\omega_1 t \pm \epsilon_1).$$

If the displacement of one of the motions is zero when $t = 0$, we have $\epsilon = \frac{\pi}{2}$; if $\epsilon_1 = \epsilon + n\pi$, we have

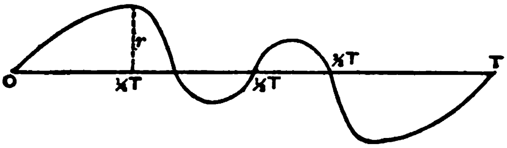
$$x = r\sin\omega t + r_1\sin(\omega_1 t + n\pi).$$

If the period of one motion is twice that of the other, for instance, we have $\omega_1 = 2\omega$, and

$$x = r\sin(\omega t) + r_1\sin(2\omega t + n\pi).$$

If the difference of phase is zero, $n = 0$; and if the amplitudes are equal also, we have

$$x = r\sin\omega t + r\sin(2\omega t).$$



This gives a curve as shown in the figure.

Periods Unequal.—We have in general

$$x = r\cos(\omega t + \epsilon), \quad y = r_1\cos(\omega_1 t + \epsilon_1)$$

for the two component rectilinear harmonic motions at right angles. The elimination of t in any case gives the curve of resultant compound harmonic motion.

If the periods of the components are as 1 to 2, and ϵ is the *difference of the epochs*, we have for equal amplitudes

$$x = r\cos(2\omega t + \epsilon), \quad y = r\cos\omega t.$$

Eliminating t,

$$x = r\left\{\left(\frac{2y^2}{r^2} - 1\right)\cos\epsilon + 2\frac{y}{r}\sqrt{1 - \frac{y^2}{r^2}}\,.\sin\epsilon\right\},$$

which is the general equation of the curve for any value of ϵ.

Thus for $\epsilon = 0$, or equal epochs,

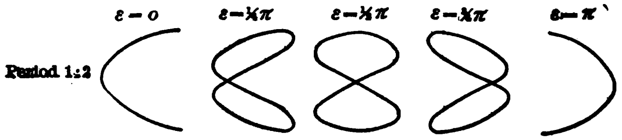
$$\frac{x}{r} = \frac{2y^2}{r^2} - 1, \quad \text{or} \quad y^2 = \frac{r}{2}\left(x + r\right),$$

which is the equation of a parabola. For $\epsilon = \frac{\pi}{2}$,

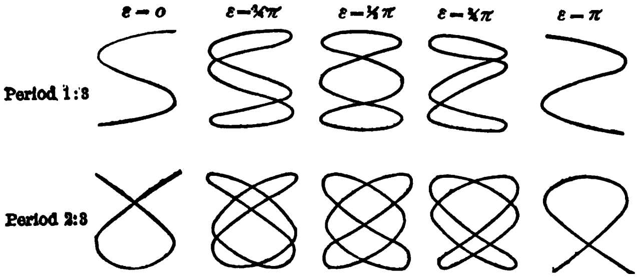
$$\frac{x}{r} = 2\frac{y}{r}\sqrt{1 - \frac{y^2}{r^2}}, \quad \text{or} \quad r^2x^2 = 4y^2(r^2 - y^2),$$

which is also the equation of a parabola.

If we make ϵ in succession, 0, 1, 2, etc., eighths of a circumference, we obtain a series of curves as shown.



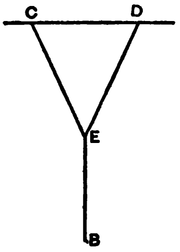
In the same way we can find the curve for any ratio of periods and difference of epoch. Thus if the periods are as 1 to 3 or 2 to 3, and we make ϵ in succession 0, 1, 2, etc., eighths of a circumference, we obtain the following series of curves:



Blackburn's Pendulum.—The motion of a pendulum which swings through a small arc is, as we shall see hereafter (page 154), simple harmonic, and the projection of the bob on a horizontal plane moves with simple rectilinear harmonic motion.

Curves similar to those just given are therefore traced by *Blackburn's pendulum.* This consists of two pendulums, *CED* and *EB*, arranged so as to swing in two planes at right angles.

Any difference of period may be made by adjusting the lengths of the pendulums, and they may be started with any difference of epoch. If the bob *B* is made to trace its path on a horizontal plane, we have, approximately, the compound harmonic curve.



[**Application of the Calculus to Harmonic Motion**.—Let a' be the known acceleration at the known distance r'. Then the acceleration at any other distance is $\frac{a'}{r'}r$, where r is the distance from the centre.

For the acceleration in the direction of the axis of X we have then

$$\frac{d^2x}{dt^2} = -\frac{a'}{r'}x; \qquad \text{. . . . .} \qquad (1)$$

and in the direction of the axis of Y,

$$\frac{d^2y}{dt^2} = -\frac{a'}{r'}y, \qquad \text{. . . . .} \qquad (2)$$

the minus sign denoting direction towards the centre.

Multiply (1) by dx and we have

$$\frac{d^2x}{dt^2}dx = -\frac{a'}{r'}xdx.$$

Integrating,

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = -\frac{a'}{2r'}x^2 + \text{Const.}$$

Let $\frac{dx}{dt} = 0$, when $x = r$, then Const. $= \frac{a'}{2r'}r^2$ and

$$\left(\frac{dx}{dt}\right)^2 = \frac{a'}{r'}(r^2 - x^2), \quad \text{or} \quad dt\sqrt{\frac{a'}{r'}} = -\frac{dx}{\sqrt{r^2 - x^2}}, \quad . \quad . \quad . \quad . \quad (3)$$

where we take the minus sign to indicate that x diminishes as t increases, or motion towards the centre.

In the same way we obtain from (2), if $\frac{dy}{dt} = 0$, when $y = r_1$,

$$dt\sqrt{\frac{a'}{r'}} = \frac{dy}{\sqrt{r_1^2 - x^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where we take the plus sign to indicate that y increases as t increases.

Integrating (3) and (4), we have

$$t\sqrt{\frac{a'}{r'}} + C = \cos^{-1}\frac{x}{r}, \quad \text{or} \quad x = r\cos\left\{t\sqrt{\frac{a'}{r'}} + C\right\}, \quad . \quad . \quad . \quad (5)$$

$$t\sqrt{\frac{a'}{r'}} + C_1 = \sin^{-1}\frac{y}{r_1}, \quad \text{or} \quad y = r_1\sin\left\{t\sqrt{\frac{a'}{r'}} + C_1\right\}, \quad . \quad . \quad . \quad (6)$$

where C and C_1 are constants of integration.

Equations (5) and (6) may be written

$$x = r\cos C\cos t\sqrt{\frac{a'}{r'}} - r\sin C\sin t\sqrt{\frac{a'}{r'}} = A_1\cos t\sqrt{\frac{a'}{r'}} + A_2\sin t\sqrt{\frac{a'}{r'}},$$

$$y = r_1\cos C_1\sin t\sqrt{\frac{a'}{r'}} + r_1\sin C_1\cos t\sqrt{\frac{a'}{r'}} = B_1\sin t\sqrt{\frac{a'}{r'}} + B_2\cos t\sqrt{\frac{a'}{r'}},$$

where $A_1 = r\cos C$, $A_2 = -r\sin C$, $B_1 = r_1\cos C_1$, $B_2 = r_1\sin C_1$.

If we find from these equations the values of sin and cos in terms of x and y and add their squares, we have, by eliminating t,

$$x^2(B_1^2 + B_2^2) + y^2(A_1^2 + A_2^2) - 2xy(A_1B_2 + A_2B_1) = (A_1B_1 - A_2B_2)^2. \quad (7)$$

This is the equation of an ellipse referred to its centre and rectangular axes.

If we take one of the principal axes corresponding with the axis of X, and count the time from the end of this axis, we have for $t = 0$, $y = 0$ and $\frac{dx}{dt} = 0$ and $x = r$. Therefore, from (5) and (6), $C = 0$ and $C_1 = 0$, and therefore $A_2 = 0$, $A_1 = r$, $B_2 = 0$, $B_1 = r_1$, and (7) becomes

$$r^2y^2 + r_1^2x^2 = r^2r_1^2,$$

or the equation of an ellipse referred to its principal axes.

We have also, from (5) and (6),

$$x = r\cos t\sqrt{\frac{a'}{r'}} \quad \text{and} \quad \frac{dx}{dt} = -r\sqrt{\frac{a'}{r'}}\sin t\sqrt{\frac{a'}{r'}},$$

$$y = r_1\sin t\sqrt{\frac{a'}{r'}} \quad \text{and} \quad \frac{dy}{dt} = r_1\sqrt{\frac{a'}{r'}}\cos t\sqrt{\frac{a'}{r'}}.$$

Therefore elliptic harmonic motion can be considered as the resultant of two simple rectilinear harmonic motions at right angles of the same period and different amplitudes, so related that the velocity of one is zero when the velocity of the other is a maximum, i.e., one is at the centre when the other is at its greatest range.

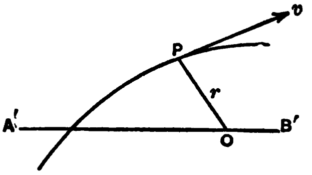
They therefore differ in epoch by 90°.

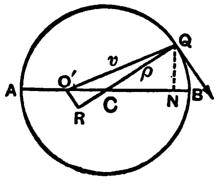
The time of a complete oscillation is for each of these component motions $2\pi\sqrt{\frac{r'}{a'}}$, and this therefore is the periodic time in the ellipse.

Planetary Motion.—Velocity Inclined to the Central Acceleration—Acceleration Inversely as the Square of the Radius Vector—Hodograph a Circle.—Since the acceleration is central, we have (page 130) $r^2\omega = c$, or $\omega = \frac{c}{r^2}$, where c is a constant and *equal to twice the areal velocity of the radius vector.* Also by assumption we have

$$f = \frac{a'r'^2}{r^2},$$

where a' is the acceleration at a known distance r'.





Let P be a point which has the velocity v, and central acceleration directed always towards the point O, the radius vector being $OP = r$.

Take O' as the pole of the *hodograph* (page 52), and draw $O'Q$ parallel and equal to v. Then the tangent to the hodograph at Q is the direction of the acceleration f at P and is parallel to $OP = r$.

Since the angular velocity ω at P is the angular velocity of the radius vector r, the angular velocity of the tangent at Q is also ω.

Let C be the centre of curvature of the hodograph, so that CQ is perpendicular to the tangent at Q and $CQ = \rho =$ radius of curvature. Then since the linear acceleration f of P is the linear velocity of Q, we have $f = \rho\omega$, or $\rho = \frac{f}{\omega}$.

But $\omega = \frac{c}{r^2}$ and $f = \frac{a'r'^2}{r^2}$. Hence $\rho = \frac{a'r'^2}{c}$. The radius of curvature ρ is therefore constant and **the hodograph for planetary motion is a circle.**

The path which, in consequence of *aberration*, a fixed star seems to describe is the hodograph of the earth's orbit, and is therefore a circle whose plane is parallel to the plane of the ecliptic.

The Path for Planetary Motion is a Conic Section. — Draw $O'R$ perpendicular to CQ and therefore parallel to r. $O'R$ is the component of the velocity v in the direction of the radius vector. Draw QN perpendicular to $O'CB$. Then QN is the component of v perpendicular to the fixed line CB.

But by similar triangles

$$\frac{O'R}{O'C} = \frac{QN}{CQ}, \quad \text{or} \quad \frac{O'R}{QN} = \frac{O'C}{CQ} = \epsilon = \text{a constant};$$

that is, the ratio of the velocity along the radius vector to the velocity at right angles to any fixed line parallel to $O'CB$ is constant and equal to ϵ.

If then r_1 and r_2 are the initial and final values of r for an indefinitely short time, and d_1, d_2 the corresponding distances of P from any given fixed line $A'B'$ parallel to $O'CB$, we have

$$\frac{O'R}{QN} = \frac{r_1 - r_2}{d_1 - d_2} = \epsilon, \quad \text{or} \quad r_1 - r_2 = \epsilon(d_1 - d_2). \quad . \quad . \quad . \quad (1)$$

Since this holds whenever we take the fixed line $A'B'$, let us take the initial distance d_1 such that $d_1 = \frac{r_1}{\epsilon}$, or $\epsilon = \frac{r_1}{d_1}$.

Then, from (1), $d_2 = \frac{r_2}{\epsilon}$, or $\epsilon = \frac{r_2}{d_2}$, and we have

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \epsilon;$$

that is, the ratio of the distance r of the moving point P from a fixed point O to its distance d from a fixed line *has a constant value.*

This is the Property of a Point Moving in a Conic Section. If $\epsilon = 1$, then $O'R = QN$, and the pole O' is on the circumference of the hodograph, and the path of P is a *parabola.*

If ϵ is less than unity, the pole O' is inside the hodograph and the path of P is an ellipse.

If ϵ is greater than unity, the pole O' is outside the hodograph and the path of P is an hyperbola.

When, therefore, a point has a central acceleration inversely proportional to the square of the distance from the centre, *it must move in a conic section with the centre of acceleration as a focus.*

Conversely, if the path be a conic section and the acceleration is directed towards either focus, it must vary inversely as the square of the distance from the focus.

In both cases the radius vector describes equal areas in equal times (page 130).

Kepler's Laws.—By laborious comparison of the observations which TYCHO BRAHE had made through many years of the planets, especially of Mars, KEPLER discovered the three laws of planetary motion which are known as KEPLER'S LAWS. He gave these laws simply as the expression of facts which seemed warranted by the observations.

The three laws are as follows:

I. *The planets describe ellipses, the sun occupying one of the foci.*

II. *The radius vector of each planet describes equal areas in equal times.*

III. The "Harmonic Law," so called. *The squares of the periods of the planets are proportional to the cubes of their mean distances from the sun.*

The second law, as we have seen (page 130), is a necessary consequence of central acceleration.

From the first law, as we have just seen, it follows that the acceleration must be inversely as the square of the distance.

The third law is a direct consequence of such central acceleration, as we shall see in the next Article.

Verification by Application to the Moon.—Assuming KEPLER'S third law, NEWTON was led directly to the conclusion that the acceleration must be inversely as the square of the distance, as follows:

The moon and other satellites move around their primaries in sensibly circular orbits, the centre being at the centre of the primary.

If T and T_1 are the periodic times of two satellites, then according to Kepler's third law, if r and r_1 are the radii of the orbits, we must have

$$\frac{T^2}{T_1^2} = \frac{r^3}{r_1^3}.$$

If ω is the angular velocity of one satellite, we have (page 76) $r\omega = \frac{2\pi r}{T}$, or $\omega = \frac{2\pi}{T}$. We have also for the acceleration (page 77) $f = \frac{v^2}{r} = r\omega^2 = \frac{4\pi^2 r}{T^2}$. For the other satellite we have in like manner $f_1 = \frac{4\pi^2 r_1}{T_1^2}$. We have then

$$\frac{f_1}{f} = \frac{T^2 r_1}{T_1^2 r} = \frac{r^2}{r_1^2},$$

or the acceleration is inversely as the square of the distance. Conversely, if the acceleration is inversely as the square of the distance, Kepler's third law is a necessary consequence.

The numerical verification of this conclusion by the moon is given in Example 17 (page 55).

[Application of the Calculus to Planetary Motion.]—The general formulas for central acceleration have been already given, Chap. VIII, page 85.

For any given law of central acceleration we have only to insert the corresponding value of $\int f dr$ in these general equations.

(*a*) **To Determine the Path when the Central Acceleration Varies Inversely as the Square of the Distance from the Pole.**—When the acceleration is inversely as the square of the distance we have

$$f = \frac{a'r'^2}{r^2},$$

where a' is the acceleration at a known distance r', and therefore $a'r'^2$ is constant. We have in this case

$$\int f dr = \int \frac{a'r'^2 dr}{r^2} = -\frac{a'r'^2}{r}.$$

Substituting this in equation (42), page 87, we have for the differential equation of the path

$$c^2\left(\frac{dr^2}{r^4 d\theta^2} + \frac{1}{r^2}\right) = c_1 + \frac{2a'r'^2}{r},$$

or, taking $d\theta$ as *always positive*,

$$d\theta = \frac{dr}{r^2\sqrt{\frac{c_1}{c^2} + \frac{2a'r'^2}{c^2 r} - \frac{1}{r^2}}}.$$

If we put $z = \frac{c^2}{r} - a'r'^2$, and $n^2 = a'^2 r'^4 + c_1 c^2$, this becomes

$$d\theta = -\frac{dz}{\sqrt{n^2 - z^2}}.$$

Integrating, we have

$$\theta = \cos^{-1}\frac{z}{n} + \text{Const.}$$

When $z = n$ and therefore $\frac{1}{r} = \frac{a'r'^2 + n}{c^2} = \frac{a'r'^2}{c^2} + \sqrt{\frac{a'^2r'^4}{c^4} + \frac{c_1}{c^2}}$, let $\theta = \phi$. Then Const. $= \phi$ and

$$\theta - \phi = \cos^{-1}\frac{z}{n}, \quad \text{or} \quad z = n\cos(\theta - \phi).$$

Replacing the values of z and n, we have for the polar equation of the path

$$r = \frac{\dfrac{c^2}{a'r'^2}}{1 + \sqrt{1 + \dfrac{c_1c^2}{a'^2r'^4}}\cos(\theta - \phi)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

This is the polar equation *of a conic section* with the pole at a focus. It will be an ellipse, parabola or hyperbola according as $c_1 \underset{>}{\overset{<}{=}} 0$.

From equation (87), page 87, since $\int f dr = -\frac{a'r'^2}{r}$, if r_1 and v_1 are the initial distance and velocity at any instant,

$$c_1 = v_1^2 - \frac{2a'r'^2}{r_1}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

and according as this is $\underset{>}{\overset{<}{=}} 0$ we have an ellipse, parabola or hyperbola. We see then that *the form of the orbit depends solely upon the magnitude of the initial velocity and not upon its direction.* Also if $a'r'^2$ is negative, that is, when the acceleration is directed *away* from the pole, *we have always an hyperbola.*

From page 100 we have seen that the speed attained by a body starting from rest at an infinite distance from the centre and moving in a straight line with an acceleration inversely as the square of the distance is $\sqrt{\frac{2a'r'^2}{r_1}}$.

Hence the orbit will be an ellipse, parabola or hyperbola according as the velocity of projection *is less than, equal to or greater than that acquired from an infinite distance.*

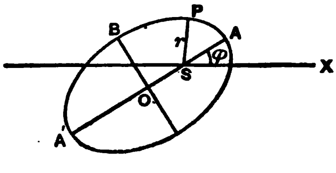
If ϵ_1 is the angle of v_1 with r_1, we have from equation (30), page 85,

$$c = r_1v_1\sin\epsilon_1. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

The constants are therefore given by (2) and (3) and the orbit is determined when the initial velocity v_1 at the distance r_1 and with the direction ϵ_1 are given.

(*b*) **Central Acceleration Inversely as the Square of the Distance from the Focus—Path an Ellipse.**—When the path is an ellipse we have the case of planetary motion.

Let the point P move in the ellipse ABA' with central acceleration always directed towards the sun S in the focus, and varying inversely as the square of the distance or radius vector $SP = r$.



Let the semi-major axis $OA = A$ and the semi-conjugate axis $OB = B$ and the eccentricity $\frac{OS}{A} = e$.

Let the angle ASX of the major axis with any fixed line SX through the focus be ϕ.

The extremity A of the major axis nearest to the focus S is called in general the *lower apsis*, or, in the case of a planet, the *perihelion*. The angle ASX is the *longitude of the perihelion*. The distance SA is the *lower apsidal distance*, or the *perihelion distance*. The other extremity A' is the *higher apsis*, or, in the case of a planet, the *aphelion*, and SA' is the *higher apsidal distance*, or *aphelion distance*.

The angle PSX, or the angle of the radius vector with the fixed line SX, we denote by θ.

The angle PSA made by the radius vector with the major axis is then $\theta - \phi$. This angle is called the *true anomaly*.

The polar equation of an ellipse with reference to the focus S as a pole, counting the angle PSA around from the perihelion, is

$$r = \frac{A(1 - e^2)}{1 + e\cos(\theta - \phi)}. \quad \dots\dots\dots\dots \quad (4)$$

But equation (1), page 142, just deduced, is the equation of a conic section, which becomes an ellipse, therefore, when

$$e^2 = 1 + \frac{c_1c^2}{a'^2r'^4}, \quad \text{and} \quad A(1 - e^2) = \frac{c^2}{a'r'^2}, \quad \text{or} \quad A = -\frac{a'r'^2}{c_1}.$$

Inserting the values of (2) and (3), we have

$$e^2 = 1 + \frac{r_1^2v_1^2\sin^2\epsilon_1\left(v_1^2 - \dfrac{2a'r'^2}{r_1}\right)}{a'^2r'^4}, \quad \dots\dots\dots \quad (5)$$

$$A = \frac{a'r'^2r_1}{2a'r'^2 - r_1v_1^2}, \quad \dots\dots\dots\dots\dots \quad (6)$$

where v_1 is the initial velocity at the distance r_1 making the angle ϵ_1 with r_1.

The elliptic orbit is thus determined.

From (6) we see that the semi-major axis A depends only on the distance r_1 and the velocity of projection v_1 and is independent of the direction of projection ϵ_1. In whatever direction the body is projected, the major axis will be the same for the same distance and velocity of projection.

We have also

$$a'r'^2 = \frac{c^2}{A(1 - e^2)}. \quad \dots\dots\dots\dots \quad (7)$$

But we know (page 85) that c is twice the areal velocity of the radius vector. If T is the periodic time, then, since twice the area of an ellipse is $2\pi A^2\sqrt{1 - e^2}$, we have

$$cT = 2\pi A^2\sqrt{1 - e^2},$$

or

$$\frac{c^2}{A(1 - e^2)} = a'r'^2 = \frac{4\pi^2A^3}{T^2}.$$

But by KEPLER's third law we have for two different planets

$$\frac{T^2}{T_1^2} = \frac{A^3}{A_1^3}, \text{ or } \frac{A^3}{T^2} = \frac{A_1^3}{T_1^3}.$$

Hence $\dfrac{c^2}{A(1-e^2)}$, or $a'r'^2$ *is constant for all the planets.*

But a' is the acceleration at a distance r', and $a'r'^2$ is equal in magnitude to the acceleration at the distance unity, since the acceleration at any distance r is $f = \dfrac{a'r'^2}{r^2}$.

The direct consequence of Kepler's third law, therefore, is that *for all the planets the acceleration is the same at the same distance from the sun.*

"Of all the laws," says Sir John Herschel, "to which induction from pure observation has ever conducted man, this third law of Kepler may justly be regarded as the most remarkable, and the most pregnant with important consequences. When we contemplate the constituents of the planetary system from the point of view which this relation affords us, it is no longer mere analogy which strikes us, no longer a general resemblance among them as individuals independent of each other, and circulating about the sun, each according to its own peculiar nature, and connected with it by its own peculiar tie. The resemblance is now perceived to be a true *family* likeness; they are bound up in one chain; interwoven in one web of mutual relation and harmonious agreement; subjected to one pervading influence, which extends from the centre to the farthest limit of that great system, of which all of them, the earth included, must henceforth be regarded as members." *

(*c*) **Value of a' for Planetary Motion.**—In all our equations for central acceleration we see that it is necessary to know the acceleration a' at some known distance r'.

We are dealing in this portion of our subject with *Kinematics*, or the study of motion only, apart from its causes or the properties of matter. It is therefore not the place here to deal with ideas of "force" and "mass."

It is sufficient to state here that the "mass" of a body is the number of standard pounds it will balance at any point of the earth's surface in an equal-armed balance. The unit of mass is then the pound.

It will be proved hereafter (see Vol. 2, Statics) that if M is the mass of the sun, and m the mass of a planet, the value of a' which must be used in all equations for planetary motion is given by

$$a' = \frac{M+m}{m'}g, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where m' is the mass of the earth, and g the mean acceleration of a body at the earth's surface due to gravity.

If the two attracting bodies are the earth and a small body of mass m, then $a' = \frac{m'+m}{m'}g$, or, since m is insignificant with respect to m', $a' = g$.

If in the preceding Article we had used the value of a' given by (1), we should have obtained

$$\frac{M+m}{m'}gr'^2 = \frac{4\pi^2A^3}{T^2} \text{ and } \frac{M+m_1}{m'}gr'^2 = \frac{4\pi^2A_1^3}{T_1^2},$$

and hence

$$T^2 : T_1^2 :: \frac{A^3}{M+m} : \frac{A_1^3}{M+m_1}.$$

We see then that Kepler's third law is not, strictly speaking, exact. The value of $\frac{a'r'^2}{r^2}$, or the acceleration at the same distance, is not, strictly speaking, constant for all the planets. The more accurate expression of the third law is that the squares of the periodic times are directly as the cubes of the semi-major axes and *inversely as the sum of the masses of the sun and planet.*

The error from this source is, however, too slight to be perceptible, the mass of Jupiter, the largest of the planets, being less than a thousandth part of that of the sun.

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* Outlines of Astronomy.

The motion of translation of the planets is not affected by their rotation on their axes, and we may treat them, therefore, as material points so far as translation is concerned.

The sun is not, strictly speaking, a fixed point in this sense, but both sun and planet move in orbits, so that the pole or focus is not at the sun's centre. and this affects the accuracy of Kepler's first two laws. The sun is also attracted by the other planets, and the planets attract each other.

The attraction of the planets for each other sensibly modifies their orbits. The ellipse is therefore only an approximation to the path, and requires correction.

Kepler's laws are thus only approximate expressions. If there were but two bodies, one fixed and the other free to move, then Kepler's first two laws would be accurate, and the third law would approach accuracy as the mass of the moving body becomes insignificant with respect to the mass of the other.

(*d*) To Find the Velocity of a Planet at any Point of its Orbit.—From equation (37), page 87, Chap. VIII, we have

$$v^2 = c_1 - 2\int f dv.$$

We have for planetary motion $\int f dv = -\frac{a'r'^2}{r}$, where a' is given in the preceding Article, and r' is the mean radius of the earth. Also from equation (2), page 142,

$$c_1 = v_1^2 - \frac{2a'r'^2}{r_1},$$

where v_1 is the velocity at the distance r_1. Therefore

$$v^2 = v_1^2 + 2a'r'^2\left(\frac{1}{r} - \frac{1}{r_1}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, since (page 143) $a'r'^2 = \frac{c^2}{A(1-e^2)}$,

$$v^2 = v_1^2 + \frac{2c^2}{A(1-e^2)}\left(\frac{1}{r} - \frac{1}{r_1}\right).$$

From page 85, $v^2 = \frac{c^2}{p^2}$, and for an ellipse, from Analytical Geometry, $p^2 = \frac{A^2(1-e^2)r}{2A-r}$. Hence

$$v^2 = \frac{c^2(2A-r)}{A^2(1-e^2)r} = \frac{a'r'^2}{Ar}(2A-r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) gives the velocity for any distance r if the semi-major axis A is known.

Equation (1) becomes the same as equation (2), page 99, for rectilinear motion.

COR. 1. We see that the velocity is greatest where r is least, or at perihelion, and least at aphelion, where r is greatest.

COR. 2. If a point moves in a circle of radius r with a speed v_1, its central acceleration is $\frac{v_1^2}{r}$ (page 58). If this acceleration is equal at any instant to the acceleration of the planet, we have from equation (7), page 143,

$$\frac{v_1^2}{r} = \frac{a'r'^2}{r^2} = \frac{c^2}{A(1-e^2)}\frac{1}{r^2}.$$

Therefore, from (2),

$$v^2 : v_1^2 :: \frac{c^2(2A - r)}{A^2(1 - e^2)r} : \frac{c^2}{A(1 - e^2)}\frac{1}{r},$$

or

$$v^2 : v_1^2 :: 2A - r : A.$$

That is, *the square of the speed in the ellipse is to the square of the speed in the circle as the distance of the planet from the unoccupied focus is to the semi-major axis.*

COR. 3. If r_1 is the perihelion distance and r_2 the aphelion distance, we have, from (2),

$$\text{for } r = r_1,\ v^2 = \frac{a'r'^2}{A}\frac{r_2}{r_1};$$

$$\text{for } r = r_2,\ v^2 = \frac{a'r'^2}{A}\frac{r_1}{r_2};$$

while for $r = A$ we have

$$v^2 = \frac{a'r'^2}{A}.$$

That is, *the speed at the extremity of the minor axis is a mean proportional between the speeds at perihelion and aphelion.*

(*e*) **To Find the Time of Describing any Portion of a Planet's Orbit.**—From equation (39), page 87, we have, when we substitute $\int f dr = -\frac{a'r'^2}{r}$,

$$\frac{dr^2}{dt^2} + \frac{c^2}{r^2} - \frac{2a'r'^2}{r} = c_1.$$

From page 143 we have $c^2 = a'r'^2A(1 - e^2)$ and $c_1 = -\frac{a'r'^2}{A}$.

Substituting these values and multiplying by $\frac{r^2A}{a'r'^2}$, we obtain

$$\frac{Ar^2dr^2}{a'r'^2dt^2} + (A - r)^2 - A^2e^2 = 0.$$

Hence

$$dt = \left(\frac{A}{a'r'^2}\right)^{\frac{1}{2}}\frac{rdr}{\sqrt{A^2e^2 - (A - r)^2}}.$$

In order to integrate this expression, let $A - r = Aez$, then

$$dt = -\left(\frac{A^3}{a'r'^2}\right)^{\frac{1}{2}}\frac{(1 - ez)dz}{\sqrt{1 - z^2}} = \left(\frac{A^3}{a'r'^2}\right)^{\frac{1}{2}}\left\{\frac{-dz}{\sqrt{1 - z^2}} + \frac{ezdz}{\sqrt{1 - z^2}}\right\}.$$

Integrating,

$$t = \left(\frac{A^3}{a'r'^2}\right)^{\frac{1}{2}}\left\{\cos^{-1}z - e(1 - z^2)^{\frac{1}{2}}\right\} + \text{Const.}$$

When $z = 1$, or $r = A - Ae$, or the planet is at perihelion, let $t = t_1$.
Then Const. $= t_1 =$ the time at perihelion.
Hence the time for any portion of the path from perihelion is

$$t - t_1 = \left(\frac{A^3}{a'r'^2}\right)^{\frac{1}{2}}\left\{\cos^{-1}z - e(1 - z^2)^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $z = \frac{A - r}{Ae}$, or $r = A(1 - ez)$.

When $z = -1$, or $r = A + Ae$, the planet is at aphelion, and $t - t_1$ is the time of half a revolution, or

$$t - t_1 = \pi\left(\frac{A^3}{a'r'^2}\right)^{\frac{1}{2}} = \frac{T}{2},$$

where T is the periodic time.

We have then

$$T^2 = \frac{4\pi^2 A^3}{a' r'^2},$$

or *the square of the periodic time varies as the cube of the semi-major axis.*

EXAMPLES.

(1) *Find the speed and periodic time of a body moving in a circle at a distance from the earth's centre of n times the earth's radius, the acceleration being inversely as the square of the distance.*

Ans. $v = \left(\frac{gr'}{n}\right)^{\frac{1}{2}}$, $T = 2\pi\left(\frac{n^3 r'}{g}\right)^{\frac{1}{2}}$.

(2) *A body at a distance r_1 from the centre of the earth is projected in a direction which makes an angle of 60° with the distance r_1, with a speed v_1 which is to the speed acquired by falling from an infinite distance as 1 to $\sqrt{3}$; the acceleration varying inversely as the square of the distance. Find the major axis, the eccentricity, the periodic time and the position of the lower apsis.*

Ans. $2A = \frac{3}{2}r_1$, $e = \frac{1}{3}$, $T = \frac{3\pi r_1}{4r'}\sqrt{\frac{3r_1}{g}}$, where r' is the radius of the earth. The lower apsis is at a distance $\frac{1}{2}r_1$ from the focus.

(3) *A body revolves about a centre, the acceleration directed towards the centre and varying directly as the distance. To determine the motion.*

From the general equations of page 86 we can determine the motion. In the present case we have $f = \frac{a'}{r'}r$, and therefore $\int f dr = \frac{a' r^2}{2r'}$.

Substituting this in equation (42), page 87, we have for the differential equation of the path

$$c^2\left(\frac{dr^2}{r^4 d\theta^2} + \frac{1}{r^2}\right) = c_1 - \frac{a'}{r'}r^2,$$

or, since $d\theta$ is always positive,

$$d\theta = \frac{dr}{r^2\sqrt{\frac{c_1}{c^2} - \frac{a' r^2}{c^2 r'} - \frac{1}{r^2}}}.$$

If we put $z = \frac{1}{r^2} - \frac{c_1}{2c^2}$, and $n^2 = \frac{c_1^2 r' - 4a'c^2}{4r'c^4}$, we have

$$d\theta = \frac{-dz}{2\sqrt{n^2 - z^2}}.$$

Integrating, we have

$$\theta = \frac{1}{2}\cos^{-1}\frac{z}{n} + \text{Const.}$$

When $z = n$, let $\theta = \phi$. Then Const. $= \phi$ and

$$(\theta - \phi) = \frac{1}{2}\cos^{-1}\frac{z}{n}, \quad \text{or} \quad z = n\cos 2(\theta - \phi).$$

Replacing the value of s, we have after reduction, for the polar equation of the path,

$$r^2 = \frac{\dfrac{2c^2}{c_1 + 2nc^2}}{1 - \dfrac{4nc^2}{c_1 + 2nc^2} \sin^2 (\theta - \phi)}.$$

This is the polar equation of an *ellipse*, the pole *at the centre of the ellipse*, where

$$A^2 = \frac{2c^2}{c_1 + 2nc^2}, \quad \frac{A^2 - B^2}{B^2} = - \frac{4nc^2}{c_1 + 2nc^2}, \text{ and hence } B^2 = \frac{2c^2}{c_1 - 2nc^2}.$$

For the values of c and c_1 we have from equations (30) and (37), page 87,

$$c_1 = v_1^2 + \frac{a'r_1^2}{r'}, \quad \text{and} \quad c = r_1 v_1 \sin \epsilon_1,$$

where v_1 is the initial velocity of projection at a given instant, r_1 the corresponding distance from the centre at that instant, and ϵ_1 the angle of v_1 with r_1.

The path is therefore fully determined.

To find the periodic time, since the area of the ellipse is πAB, and since c is twice the area described in a unit of time by the radius vector, we have

$$cT = 2\pi AB, \quad \text{or} \quad T = \frac{2\pi AB}{c}.$$

Inserting the values of A and B, we have

$$T = \frac{4\pi c}{\sqrt{c'^2 - 4n^2c^4}}.$$

Inserting the value of n^2, we have

$$T = \frac{2\pi}{\sqrt{\dfrac{a'}{r'}}}.$$

This is the same result found in page 107 for rectilinear harmonic motion.

(4) *A particle describes an ellipse under the action of central acceleration, directed to the centre of the ellipse. To determine the law of acceleration.*

The polar equation to the ellipse, centre pole, is

$$\frac{1}{r^2} = \frac{1}{A^2} + \left(\frac{1}{B^2} - \frac{1}{A^2}\right) \sin^2 \theta,$$

from which we have

$$\frac{1}{r^2} - \frac{1}{A^2} = \left(\frac{1}{B^2} - \frac{1}{A^2}\right) \sin^2 \theta, \quad \text{and} \quad \frac{1}{r^2} - \frac{1}{B^2} = -\left(\frac{1}{B^2} - \frac{1}{A^2}\right) \cos^2 \theta.$$

Differentiating, we have

$$- \frac{dr}{r^3 d\theta} = \left(\frac{1}{B^2} - \frac{1}{A^2}\right) \cos \theta \sin \theta,$$

and hence

$$\frac{dr^2}{r^6 d\theta^2} = \left(\frac{1}{B^2} - \frac{1}{A^2}\right)^2 \cos^2 \theta \sin^2 \theta = -\left(\frac{1}{r^2} - \frac{1}{A^2}\right)\left(\frac{1}{r^2} - \frac{1}{B^2}\right).$$

Differentiating again,

$$-\frac{d^2r}{r^3d\theta^2}+\frac{3dr^2}{r^4d\theta^2}+\frac{drd^2\theta}{r^3d\theta^3}=\left(\frac{1}{B^2}-\frac{1}{A^2}\right)\left(\cos^2\theta-\sin^2\theta\right).$$

But from equation (49), page 88, we have $d^2\theta = -\frac{2drd\theta}{r}$. Therefore

$$-\frac{d^2r}{r^3d\theta^2}+\frac{dr^2}{r^4d\theta^2}=-\frac{2}{r^2}+\frac{1}{A^2}+\frac{1}{B^2}.$$

Now from equation (45), page 88, we have

$$f=c^2r\left(\frac{1}{r^4}-\frac{d^2r}{r^5d\theta^2}\right)$$

$$=c^2r\left[\frac{1}{r^4}-\frac{dr^2}{r^6d\theta^2}+\frac{1}{r^2}\left\{-\frac{d^2r}{r^3d\theta^2}+\frac{dr^2}{r^4d\theta^2}\right\}\right]$$

$$=c^2r\left[\frac{1}{r^4}+\left(\frac{1}{r^2}-\frac{1}{A^2}\right)\left(\frac{1}{r^2}-\frac{1}{B^2}\right)-\frac{2}{r^4}+\frac{1}{A^2r^2}+\frac{1}{B^2r^2}\right]=\frac{c^2r}{A^2B^2}$$

The law of acceleration is therefore *that of the direct distance.*

(5) *A particle describes a conic section under the action of central acceleration directed to one of the foci. To find the law of acceleration.*

The polar equation of either the ellipse or hyperbola, focus pole, is

$$r=\frac{\pm A(1-e^2)}{1+e\cos\theta},\quad\text{or}\quad r+er\cos\theta=\pm A(1-e^2).$$

Differentiating,

$$dr+e\cos\theta dr-er\sin\theta d\theta=0.$$

Differentiating again,

$$d^2r+e\cos\theta d^2r-e\sin\theta d\theta dr-e\sin\theta d\theta dr-er\cos\theta d\theta^2-er\sin\theta d^2\theta=0.$$

But from eq. (49), page 88, $d^2\theta=-\frac{2drd\theta}{r}$. Therefore

$$d^2r+e\cos\theta dr^2=er\cos\theta d\theta^2,\quad\text{or}\quad\frac{d^2r}{d\theta^2}=\frac{er\cos\theta}{1+e\cos\theta}.$$

Substituting this in eq. (45), page 88, we have

$$f=\frac{c^2}{r^2}\left(\frac{1}{r}-\frac{e\cos\theta}{r(1+e\cos\theta)}\right)=\frac{c^2}{r^2}\frac{1}{r(1+e\cos\theta)}=\frac{c^2}{\pm A(1-e^2)r^2}.$$

The acceleration is therefore *inversely as the square of the distance* for either ellipse or hyperbola.

The polar equation of the parabola is

$$r=\frac{2A}{1-\cos\theta},\quad\text{or}\quad r-r\cos\theta=2A.$$

Differentiating, we have

$$dr-dr\cos\theta+r\sin\theta d\theta=0.$$

Differentiating again,

$$d^2r-d^2r\cos\theta+dr\sin\theta d\theta+dr\sin\theta d\theta+r\cos\theta d\theta^2+r\sin\theta d^2\theta=0.$$

But from eq. (49), page 88, $d^2\theta=-\frac{2drd\theta}{r}$; therefore

$$d^2r-d^2r\cos\theta=-r\cos\theta d\theta^2,\quad\text{or}\quad\frac{d^2r}{d\theta^2}=-\frac{r\cos\theta}{1-\cos\theta}.$$

Substituting this in eq. (45), page 88, we have

$$f = \frac{c^2}{r^2}\left(\frac{1}{r} + \frac{\cos\theta}{r(1 - \cos\theta)}\right) = \frac{c^2}{r^2}\,\frac{1}{r(1 - \cos\theta)} = \frac{c^2}{2Ar^2}.$$

The acceleration is therefore *inversely as the square of the distance* for the parabola.

(6) *A particle describes a hyperbolic spiral under the action of central acceleration directed to the pole. To find the law of acceleration.*

The equation of the hyperbolic spiral, centre pole, is $r\theta = A$. We have then $\theta dr + rd\theta = 0$, $\theta d^2r + drd\theta + drd\theta + rd^2\theta = 0$.

From eq. (49), page 88, $d^2\theta = -\frac{2drd\theta}{r}$. Therefore $\theta d^2r = 0$, or $d^2r = 0$.

From eq. (45), page 88, we have then

$$f = \frac{c^2}{r^3},$$

or the acceleration is *inversely as the cube of the distance.*

(7) *A particle describes the lemniscate of Bernoulli under central acceleration, the centre being the node. To find the law of acceleration.*

The perpendicular from the node on the tangent is $p = \pm \frac{r^3}{2A^2}$. Hence

$$\frac{dp}{dr} = \frac{3r^2}{2A^2}, \quad \text{and} \quad \frac{dp}{p^3dr} = \frac{12A^4}{r^7}.$$

We have from eq. (48), page 88, therefore,

$$f = \frac{12c^2A^4}{r^7}.$$

(8) *A particle describes a circle under central acceleration directed to a point in the circumference. Find the law of acceleration.*

The polar equation of the circle is $r = 2R\cos\theta$. Therefore

$$dr = -2R\sin\theta d\theta, \quad \text{and} \quad d^2r = -rd\theta^2 - 2R\sin\theta d^2\theta.$$

But from eq. (49), page 88,

$$d^2\theta = -\frac{2drd\theta}{r} = \frac{4R\sin\theta d\theta^2}{r}.$$

Hence

$$d^2r = -rd\theta^2 - \frac{8R^2\sin^2\theta d\theta^2}{r} = rd\theta^2 - \frac{8R^2d\theta^2}{r},$$

and

$$\frac{d^2r}{d\theta^2} = r - \frac{8R^2}{r}.$$

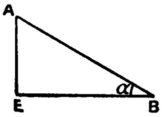
Substituting in eq. (45), page 88, we have

$$f = \frac{8R^2c^2}{r^5}.$$

CHAPTER V.

CONSTRAINED MOTION OF A POINT. SIMPLE PENDULUM. MOTION ON A CYCLOID. MISCELLANEOUS PROBLEMS.

Motion on an Inclined Plane — Uniform Acceleration.—Let a point have a uniform acceleration f in the direction AE, and let the point be constrained to move in the straight line AB which makes the angle α with the horizon.



The component of the acceleration in the direction of the motion is then $f \sin \alpha$.

The motion along AB is then rectilinear motion under uniform acceleration $f \sin \alpha$, and equations (1) to (6), page 93, apply directly, if we substitute $f \sin \alpha$ in place of g.

If v_1 is the initial velocity at A and v is the velocity at B, we have from (5), page 93,

$$v^2 - v_1^2 = 2fl \sin \alpha,$$

where l is the length of the inclined plane AB. But $l \sin \alpha = AE$.

The speed, therefore, gained in moving from A to B *is equal to that which would be gained in falling through AE with the uniform acceleration f.*

The time in falling from A to E is from (1), page 93, $t' = \frac{v - v_1}{f}$, and in passing from A to B, $t = \frac{v - v_1}{f \sin \alpha}$. Hence

$$\frac{t}{t'} = \frac{l}{AE},$$

or the times are proportional to the distances l and AE.

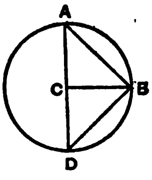
The distance passed through along AB is from (3), page 93,

$$l = v_1 t + \frac{1}{2} f \sin \alpha \,.\, t^2,$$

where v_1 is the initial velocity.

If the point starts from rest, we have for the distance along AB

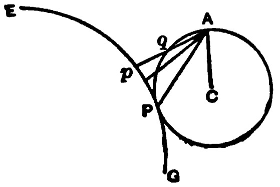
$$l = \frac{1}{2} f \sin \alpha \,.\, t^2.$$



Let AD be the vertical diameter of a circle and $AB = l$ any chord. Join DB. Then we have $AB = AD \cos DAB = AD \sin ABC$. If $AB = l$, we have also $AB = \frac{1}{2} ft^2 \sin ABC$. Therefore $AD = \frac{1}{2} ft^2$, or $t = \sqrt{\frac{2AD}{f}}$. This is independent of the position of the chord AB, and therefore *t is the same for any chord* through A or D.

Hence for uniform acceleration f, *the time of descent down all chords through the highest and lowest points of a circle are equal.*

This property enables us to find the line of swiftest descent to a given curve from any point in the same vertical plane.



Thus if EG is the curve and A the point, draw AC vertical or parallel to the direction of f, and with centre in AC describe a circle passing through A and tangent to the curve EG at P.

Then AP is the line of swiftest descent from A to the curve EG. For any other point p in EG, Ap cuts the circle in some point q, and since the time from A to q is equal to that from A to P, the time from A to p is greater.

EXAMPLES.

$g = 32.16$ ft.-per-sec. per sec. Friction, etc., disregarded.

(1) *Find the position of a point on the circumference of a vertical circle, in order that the time of rectilinear descent from it to the centre may be the same as the time of descent to the lowest point. Acceleration due to gravity.*

Ans. 30° from the top.

(2) *The straight line down which a particle will slide, under the action of gravity, in the shortest time from a given point to a given circle in the same vertical plane, is the line joining the point to the upper or lower extremity of the vertical diameter, according as the point is within or without the circle.*

(3) *Find the line of quickest descent from the focus to a parabola whose axis is vertical and vertex upwards, and show that its length is equal to that of the latus rectum. Acceleration vertical and uniform.*

(4) *Find the straight line of swiftest descent from the focus of a parabola to the curve when the axis is horizontal. Acceleration vertical and uniform.*

(5) *The times in which heavy particles slide from rest down inclined planes of equal height are proportional to their lengths.*

(6) *Show that if a circle be drawn touching a horizontal straight line in a point P and a given curve in a point Q below P, PQ is the line of swiftest descent to the curve, under constant vertical acceleration.*

(7) *Find the straight line of quickest descent from a given point to a given straight line, the point and the line being in the same vertical plane. Acceleration constant and vertical.*

Ans. From P, the given point, draw a horizontal line meeting the given line in C. Lay off along the given line CD equal to PC. PD is the required line of swiftest descent.

(8) *A given point P is in the same plane with a given vertical circle and outside it, the highest point Q of the circle being below P. Find the line of slowest descent from P to the circle. Acceleration constant and vertical.*

Ans. Join PQ and produce it to meet the circumference in R. PR is the line required.

(9) *A number of heavy particles start without velocity from a common point and slide down straight lines in various directions. Show that the locus of the points reached by them with a given speed is a horizontal plane, and that of the points reached by them in a given time is a sphere whose highest point is the starting-point.*

(10) *The times required by heavy particles to descend in straight lines from the highest point in the circumference of a vertical circle to all other points in the circumference are the same.*

Also to descend in straight lines to the lowest point in the circumference, from all other points in the circumference, the times are the same.

(11) *If heavy particles slide down the sides of a right-angled triangle whose hypothenuse is vertical, they will acquire speeds proportional to the sides.*

(12) *A point having a constant acceleration of 24 ft.-per-sec. per sec. is constrained to move in a direction in which its speed changes in 1 minute from 10 to 250 yards per sec. Find the inclination of its direction of motion to that of the given acceleration.*

Ans. 60°.

(13) *A heavy particle is projected up an inclined plane whose inclination to the horizon is 30°. Find the distance traversed during a change of speed from 48 to 16 ft. per sec.*

Ans. 63.68 ft.

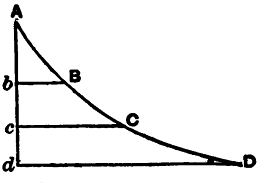
(14) *A point has, when 1 mile up an incline of 1 in 50 (i.e., one having an inclination to the horizon of* $\sin^{-1}\frac{1}{50}$*), an upward velocity of 30 miles an hour. (a) In what time will it come to a standstill? (b) If it afterwards slides down, with what speed will it reach the foot of the incline?*

Ans. (*a*) 1 min. 8.4 sec. (*b*) 68.7 miles per hour.

(15) *A body slides from rest down a smooth inclined plane and then falls to the ground. The length of the plane is 18 ft., its inclination to the horizon 30°, and the height of its lowest point from the ground 40 ft. Find the distance horizontally from the end of the plane to the point where the body reaches the ground.* [Take $g = 32$ ft.-per-sec. per sec.]

Ans. $15\sqrt{3}$ ft.

Motion in a Curved Path—Uniform Acceleration.—Let *ABCD* be any curved path, and *Ad* the direction of the acceleration *f*. Any very small portion of the curve, *AB*, may be considered as a straight line. We have then, as on page 151, the change of speed in moving from *A* to *B*, the same as in moving from *A* to *b* with the constant acceleration *f*. So also, in moving from *B* to *C*, the change of speed is the same as in moving from *b* to *c* with the constant acceleration *f*.

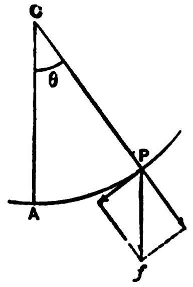


Hence the change of speed in traversing any portion of the path *AD* is the same as in traversing with constant acceleration *f* the projection *Ad* of the path on a line in the direction of the acceleration.

If then v_1 is the initial speed at A, and v is the speed at any point D, we have

$$v^2 - v_1^2 = 2f\,.\,\overline{Ad}.$$

Motion in a Circle—Uniform Acceleration.—This is the case of the *simple pendulum*, which consists of a heavy particle attached to a fixed point by a massless inextensible string.



Let C be the point of suspension and CA the radius, and let the acceleration f be uniform and vertical. For any position of the point P the angle $ACP = \theta$, and the acceleration may be resolved into a tangential component $f \sin \theta$ and into a normal component $f \cos \theta$.

The normal component has no effect upon the motion in the curve at P.

If the angle θ is very small, the arc will not differ materially from the sine, and we have $\sin \theta = \frac{\text{arc } AP}{l}$, where l is the length of the radius CA.

The tangential acceleration at the point P is then $a = \frac{f \times \text{arc } AP}{l}$. It is therefore directly proportional to the displacement of P from A, measured along the path.

The motion of P is thus *simple harmonic* motion about A as a centre (page 103).

The periodic time is then (page 104)

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{tangential acceleration}}} = 2\pi\sqrt{\frac{\text{arc } AP}{\frac{f \times \text{arc } AP}{l}}} = 2\pi\sqrt{\frac{l}{f}},$$

or for the simple pendulum the time of a *vibration* is $t = \pi\sqrt{\frac{l}{g}}$.

The periodic time is therefore *for small displacements* independent of the amplitude, and therefore for small arcs the oscillations are *isochronous*.

The time of oscillation is usually taken as half the periodic time, or the time between the instants at which the pendulum reaches opposite ends of its swing. Thus the seconds pendulum makes a complete oscillation in 2 seconds.

If θ is not very small the time is different, but the variation is practically very slight. (See page 160.)

Cor. If the velocity of P at any instant is not wholly in the plane PCA, it may be resolved into two components, one in the plane PCA and the other perpendicular to it, and both tangential to a spherical surface. Hence, in the case in which θ is small, P's motion may be resolved into two simple harmonic motions of the same period; and its motion is therefore (page 135) elliptic harmonic motion, the period being the common period of the components. The ellipse described will depend upon the amplitude and epoch of the components and therefore upon the magnitude and direction of the initial velocity of P.

If θ is not very small, and the component motions are of different amplitudes, the periods will have different values, and the point P describes curves similar to those given on page 137.

If the component motions are equal in amplitude and therefore in period and differ in phase by one quarter period, the point P moves (page 134) in a circle about the foot of the perpendicular on CA as a centre. This is the case of the conical pendulum.

EXAMPLES.

(1) *Find the time of oscillation of a pendulum* 10 *ft. long at a place at which* $g = 32$ *ft.-per-sec. per sec.*

Ans. 1.75 sec.

(2) *Find the length of the seconds pedulum at a place at which* $g = 31.9$.

Ans. 3.232 ft.

(3) *Find the length of the pendulum which makes* 24 *beats in* 1 *min. when* $g = 32.2$.

Ans. 20.39 ft.

(4) *A seconds pendulum is lengthened* 1 *per cent. How much does it lose per day?*

Ans. 7 min. 8.8 sec.

(5) *The length of the seconds pendulum being* 99.414 *cm., find the value of* g.

Ans. 981.17 cm.-per-sec. per sec.

(6) *A pendulum* 37.8 *inches long makes* 182 *beats in* 3 *min. Find the value of* g.

Ans. 31.78 ft.-per-sec. per sec.

(7) *If two pendulums at the same place make* 25 *and* 30 *oscillations respectively in* 1 *sec., what are their relative lengths?*

Ans. 1.44 to 1.

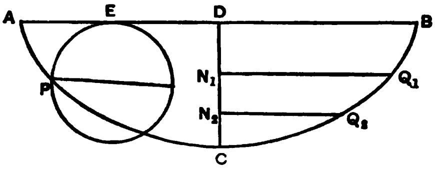
(8) *A pendulum which beats seconds at one place is carried to another where it gains* 2 *sec. per day. Compare the values of* g *at the two places.*

Ans. As 0.999953 to 1.

(9) *A pendulum which beats seconds at the sea-level is carried to the top of a mountain where it loses* 40.1 *sec. per day. Assuming the value of* g *to be inversely proportional to the distance from the centre of the earth, and the sea-level to be* 4000 *miles from that point, find the height of the mountain.*

Ans. 1.86 miles.

Motion in a Cycloid—Uniform Acceleration.—A cycloid is the curve traced by a point in the circumference of a circle which rolls allong a straight line.



If the circle EP rolls along the line AB, the point P being originally at A, the path of P is the cycloid ACB.

If C is the position of P when the diameter of the circle through P is perpendicular to AB, the line CD perpendicular to AB is the axis and C is the vertex of the cycloid.

Let the uniform acceleration f be always parallel to DC and vertical.

Let the moving point Q have at Q_1 a speed zero. Its speed at Q_2 is then (page 154)

$$v^2 = 2f . N_1N_2.$$

Let t be the time in which the point would with the same acceleration and with initial speed zero move from D to C. Then $CD = \frac{1}{2}ft^2$. Hence

$$v^2 = \frac{4}{t^2} . N_1N_2 . CD = \frac{4}{t^2}CD(CN_1 - CN_2).$$

Now by a property of the cycloid

$$4CD . CN_1 = CQ_1^2 \quad \text{and} \quad 4CD . CN_2 = CQ_2^2.$$

Hence

$$v^2 = \frac{1}{t^2}(CQ_1^2 - CQ_2^2).$$

Now $t^2 = \frac{2CD}{f}$ is a constant. Hence the motion of Q in the cycloid is simple harmonic (page 103), where $\frac{1}{t^2} = \frac{a'}{s'}$, a' being the tangential acceleration of Q at the distance s' measured along the curve. If T is the time of a complete oscillation, we have

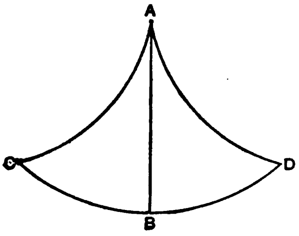
$$T = 2\pi\sqrt{\frac{s'}{a'}} = 2\pi t = 2\pi\sqrt{\frac{2CD}{f}}.$$

If t' is the time occupied in moving from Q_1 to C,

$$t' = \frac{\pi}{2}\sqrt{\frac{2CD}{f}},$$

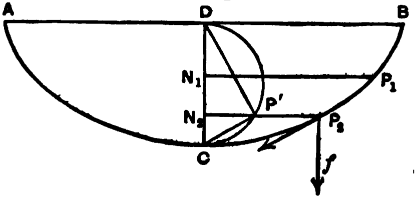
or the time of a pendulum whose length is $2CD$, or 4 times the radius of the generating circle.

As this involves only constant quantities, the time is the same whatever be the position of the starting-point Q_1, or the oscillations are isochronous. Hence the cycloid is called a *tautochrone*.



This result is rendered of practical importance by one of the properties of the cycloid, viz., that if a flexible and inextensible string AB is fixed at the end A and wrapped tightly round the semi-cycloid AC, the end B of the string as it unwinds will describe another semi-cycloid. If then AC and AD are fixed semi-cycloids, symmetrical with reference to the vertical AB, and AB is a simple pendulum, B will describe a cycloid, and its oscillations will be isochronous whatever their extent.

[**Application of the Calculus.—To Determine the Motion of a Point Constrained to Move in a Cycloid, the Acceleration being Constant, in the Direction of the Axis and towards the Vertex.**]—By the application of the general formulas of page 88 we can deduce the results already obtained.



Let the axis $CD = 2r$, where r is the radius of the generating circle $DP'C$. Let the acceleration f act downward. Let $CN = y$, $NP = x$ and the length of arc $CP = s$. Let the initial position be P_1 at the height $CN_1 = h$ above C, and the speed at P_1 be $v_1 = 0$.

We have thus the case of equation (55), page 89, and obtain at once, since f is negative and $s_1 = h$,

$$v = \sqrt{2f(h - y)}$$

for the speed at any point given by $CN = y$. When $y = 0$, we have, at the lowest point C, $v = \sqrt{2fh}$, which is the same as that due to the vertical height h.

By the property of the cycloid we have

$$s = \text{arc } CP = 2\sqrt{DC \times CN} = 2\sqrt{2ry} = 2 \text{ chord } CP'.$$

Hence

$$ds = \pm\, dy\sqrt{\frac{2r}{y}}.$$

We substitute the minus value in equation (56), page 89, because for descent the arc decreases as the time increases. We have then

$$dt = -\sqrt{\frac{r}{f}}\,\frac{dy}{\sqrt{hy - y^2}}.$$

Integrating, since for $t = 0$, $y = h$, we have

$$t = \sqrt{\frac{r}{f}}\cdot\left(\pi - \text{versin}^{-1}\frac{2y}{h}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the time of descent to the lowest point where $y = 0$, or for the time of one quarter of a complete oscillation,

$$t = \pi\sqrt{\frac{r}{f}} = \frac{\pi}{2}\sqrt{\frac{4r}{f}}.$$

The periodic time is then

$$T = 2\pi\sqrt{\frac{4r}{f}},$$

or the same as a simple pendulum (page 154) whose length is 4 times the radius of the generating circle $DP'C$.

The time is independent of h and is the same no matter what the position from which the point begins to descend. The oscillations are therefore *isochronous* and hence the cycloid is called the *tautochrone.*

The reason of this remarkable property is easily seen by considering the tangential acceleration.

In the cycloid the chord CP' is always parallel to the tangent TP. The tangential acceleration or tangential component of f is then

$$\frac{d^2s}{dt^2} = f\sin TPf = f\sin CDP' = f\frac{CP'}{CD} = f\frac{s}{4r}.$$

The tangential acceleration is therefore directly proportional to the distance from the vertex measured along the path, and the motion of P is *simple harmonic* (page 108).

The periodic time is then (page 104)

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{tangential acceleration}}} = 2\pi\sqrt{\frac{s}{f\frac{s}{4r}}} = 2\pi\sqrt{\frac{4r}{f}},$$

as already found.

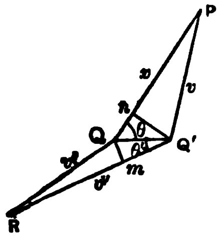
If in (1) we make $y = \frac{h}{2} = \frac{1}{2}CN_1$, we have $t = \frac{\pi}{2}\sqrt{\frac{r}{f}}$, or half the time from P_1 to C. The time, therefore, in descending through half the vertical space to C is half the time of passing from P_1 to C.

[To Find a Curve such that a Point Moving on it under the Action of Gravity will Pass from any one Given Position to any Other in Less Time than by any Other Curve through the Same Two Points.]—This is the celebrated problem of the "curve of swiftest descent" first propounded by Bernoulli. The following is founded upon his original solution.

If the time of descent through the entire curve is a minimum, that through any portion of the curve is a minimum.

It is also obvious that *between any two contiguous equal values of a continuously varying quantity, a maximum or minimum must lie.*

This principle though simple is of very great power, and often enables us to solve problems of maxima and minima, such as require not merely the processes of the Differential Calculus but those of the Calculus of Variations. The present case is a good example.



Let, then, PQ, QR and PQ', $Q'R$ be two pairs of indefinitely small sides of a polygon such that the time of descending through either pair, starting from P, may be *equal*. Let QQ' be horizontal and indefinitely small compared with PQ and QR. The curve of swiftest descent must lie *between* these paths, and must possess any property which they have in common. Hence if we draw Qm, $Q'n$ perpendicular to RQ', PQ, and let v be the speed down PQ or PQ' (supposed uniform) and v' that down QR or $Q'R$, we have for the time from P to R by either path

$$\frac{PQ}{v} + \frac{QR}{v'} = \frac{PQ'}{v} + \frac{Q'R}{v'}, \text{ or } \frac{PQ - PQ'}{v} = \frac{Q'R - QR}{v'},$$

or

$$\frac{Qn}{v} = \frac{Q'm}{v'}.$$

Now if θ be the inclination of PQ to the horizon, and θ' that of QR, we have $Qn = QQ' \cos\theta$, $Q'm = QQ' \cos\theta'$. Hence

$$\frac{\cos\theta}{v} = \frac{\cos\theta'}{v'}.$$

This is true for any two consecutive elements of the required curve, and therefore throughout the curve we have, at any point, v proportional to the cosine of the angle which the tangent to the curve at that point makes with the horizontal. But v^2 is proportional to the vertical distance h fallen through.

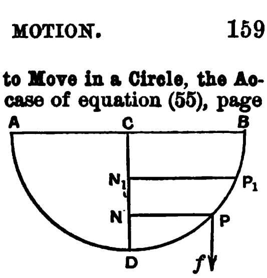
Hence the curve required is such that the cosine of the angle it makes with the horizontal line through the point of departure varies as the square root of the distance from that line.

Now in the figure of page 157 we have, from the property of a cycloid,

$$\cos CP'N = \cos TPN = \cos ODP' = \frac{DP'}{DO} = \sqrt{\frac{DN}{DO}}.$$

The curve required is therefore the cycloid. The cycloid has received on account of this property the name of *Brachistochrone.*

[To Determine the Motion of a Point Constrained to Move in a Circle, the Acceleration being Constant and Vertical.]—This is the case of equation (55), page 89.



Let $DN = y$, $NP = x$. Let the speed at P_1 be $v_1 = 0$, the distance $DN_1 = h$.

For the speed at any other point P we have at once from equation (55), page 89, since f_y is minus and $s_1 = h$,

$$v = \sqrt{2f(h-y)},$$

where y is the distance DN. When $y = 0$, we have for the lowest point, D,

$$v = \sqrt{2fh},$$

which is the same as that due to the vertical height h.

From equation (56), page 89, we have

$$dt = \frac{ds}{\sqrt{2f(h-y)}}.$$

The equation of the circle referred to D as origin is

$$x^2 = 2rx - y^2,$$

where r is the radius. Hence

$$dx^2 = \frac{(r-y)^2 dy^2}{2ry - y^2}.$$

$$\text{But } ds^2 = dx^2 + dy^2 = dy^2\left(1 + \frac{(r-y)^2}{2ry - y^2}\right) = \frac{r^2 dy^2}{2ry - y^2}.$$

$$\therefore ds = \pm \frac{r\,dy}{\sqrt{2ry - y^2}}.$$

We substitute the minus value in equation (56), page 89, because the arc decreases as the time increases, and obtain

$$dt = -\frac{r}{\sqrt{2f}} \cdot \frac{dy}{\sqrt{(h-y)(2ry-y^2)}} = -\frac{r}{\sqrt{2f}} \frac{dy}{\sqrt{(hy-y^2)(2r-y)}}. \qquad (1)$$

If y is small in comparison to $2r$, this reduces to

$$dt = -\frac{1}{2}\sqrt{\frac{r}{f}} \cdot \frac{dy}{\sqrt{hy - y^2}}.$$

Integrating, since when $t = 0$, $y = h$, we have

$$t = \frac{1}{2}\sqrt{\frac{r}{f}}\left(\pi - \operatorname{versin}^{-1}\frac{2y}{h}\right).$$

When $y = 0$,

$$t = \frac{\pi}{2}\sqrt{\frac{r}{f}}.$$

This is the time of one quarter of a complete oscillation. The periodic time is then

$$T = 2\pi\sqrt{\frac{r}{f}}.$$

This is the same result as on page 154.

If, however, y is not small in comparison to $2r$, we have, from (1),

$$dt = -\frac{r}{\sqrt{2f}} \frac{(2r-y)^{-\frac{1}{2}} dy}{\sqrt{hy - y^2}} = -\frac{1}{2}\sqrt{\frac{r}{f}} \frac{\left(1 - \frac{y}{2r}\right)^{-\frac{1}{2}} dy}{\sqrt{hy - y^2}}.$$

Expanding $\left(1 - \frac{y}{2r}\right)^{-\frac{1}{2}}$ by the binomial theorem, we have

$$dt = -\frac{1}{2}\sqrt{\frac{r}{f}} \cdot \frac{dy}{\sqrt{hy - y^2}} \left\{ 1 + \frac{1}{2}\left(\frac{y}{2r}\right) + \frac{1 . 3}{2 . 4}\left(\frac{y}{2r}\right)^2 + \frac{1 . 3 . 5}{2 . 4 . 6}\left(\frac{y}{2r}\right)^3 + \text{etc.} \right\}.$$

Thus the terms to be integrated are of the form $-\frac{y^n\,dy}{\sqrt{hy - y^2}}$, the exponents n being the natural numbers beginning with zero. Performing the integrations, and taking the limits $y = h$ and $y = 0$, we have for the time of one quarter oscillation

$$t = \frac{\pi}{2}\sqrt{\frac{r}{f}} \left\{ 1 + \left(\frac{1}{2}\right)^2\left(\frac{h}{2r}\right) + \left(\frac{1 . 3}{2 . 4}\right)^2\left(\frac{h}{2r}\right)^2 + \left(\frac{1 . 3 . 5}{2 . 4 . 6}\right)^2\left(\frac{h}{2r}\right)^3 + \text{etc.} \right\}.$$

This series converges more rapidly as h becomes smaller. If we take only the first term we have, as already found, $t = \frac{\pi}{2}\sqrt{\frac{r}{f}}$.

We can put $h = r - r\cos\theta = 2r\sin^2\frac{\theta}{2}$, where θ is the semi-angle of oscillation DCP_1; or taking the arc as equal to its sine, we have for the first two terms

$$t = \frac{\pi}{2}\sqrt{\frac{r}{f}}\left(+ \frac{\theta^2}{16}\right).$$

Thus if the point swings through an arc of $\frac{1}{10}$ radian or 5°.7, on each side of the vertical, the time of an oscillation is increased by about $\frac{1}{1600}$, or in the case of the seconds pendulum the time of a beat is increased by $\frac{1}{3200}$ of a second. For a swing on either side of the vertical of any amount we have for the time of a quarter oscillation

$$t = \frac{\pi}{2}\sqrt{\frac{r}{f}}\left(1 + \frac{1}{4}\sin^2\frac{\theta}{2} + \frac{9}{64}\sin^4\frac{\theta}{2} + \frac{225}{2304}\sin^6\frac{\theta}{2},\ \text{etc.}\right).$$

For a swing of 60° on either side we have, therefore,

$$t = \frac{\pi}{2}\sqrt{\frac{r}{f}}\left(1 + \frac{1}{16} + \frac{9}{1024}\right),$$

or the time is increased by about $\frac{1}{16}$.

MISCELLANEOUS PROBLEMS.

We give here as exercises for the student a number of problems covering the preceding Chapters. Resistance of the air and friction are neglected.

(1) *A courier travels at the rate of* 5⅛ *miles in* 8 *hours. Six hours after his departure another courier is dispatched from a place* 12 *miles behind the starting-point of the first. The second courier travels at the rate of* 8 *miles in* 10 *hours over the same route as the first. How long before he will overtake the first?*

Ans. 120 hours.

(2) *A man walks on the deck of a vessel from bow to stern at the rate of* 3 *miles an hour, while the vessel moves at the rate of* 7 *miles an hour. Find the speed of the man with reference to the earth's surface.*

Ans. 4 miles an hour, in the direction of the ship's motion.

(3) *A point moves in a given path for* 10 *seconds with a uniform rate of change of speed of* + 8 *ft.-per-sec. per sec. Find the final speed and the space traversed, if the point starts from rest.*

Ans. $v = at$, $s = \frac{1}{2}at^2$;

$v = 80$ ft. per sec., $s = 400$ ft.

(4) *A point has an initial speed of* 7.7 *ft. per sec. and a uniform rate of change of speed of* + 5.5 *ft.-per-sec. per sec. Find the time of passing over* 2200 *ft.*

Ans. $s = v_1 t + \frac{1}{2}at^2$. $t = 26.9$ sec.

(5) *A point has an initial speed of* 7 *ft. per sec. and a final speed of* 125 *ft. per sec., and describes a distance of* 3250 *ft. What is the uniform rate of change of speed?*

Ans. $s - s_1 = \frac{v^2 - v_1^2}{2a}$, $a = 2.4$ ft.-per-sec. per sec.

(6) *A point has an initial speed of* 100 *ft. per sec. and a rate of change of speed of* 1 *ft.-per-sec. per sec. Its final speed is* 7 *ft. per sec. Find the time of motion and the space described.*

Ans. $v = v_1 - at$, $t = 93$ sec.;

$s - s_1 = v_1 t - \frac{1}{2}at^2$, $s - s_1 = 4975.5$ ft.

(7) *A locomotive has a speed of* 30 *miles an hour on a level when the brakes are applied. The loss of speed due to the brakes is* 8 *ft.-per-sec. per sec. Find* (*a*) *the speed at the end of* 3 *seconds and the distance traversed;* (*b*) *the time of coming to rest;* (*c*) *the retardation in order that the locomotive may come to rest in* 30 *seconds.*

Ans. $v = v_1 - at$, $v_1 = 44$ ft. per sec.; $a = 8$ ft.-per-sec. per sec.;

$s - s_1 = v_1 t - \frac{1}{2}at^2$.

(*a*) 20 ft. per sec., 118 ft.; (*b*) $t = 5.5$ sec.; (*c*) 1.47 ft.-per-sec. per sec.

(8) *A body falls for* 4 *seconds in vacuo. Find the final velocity and the space described* ($g = 32\frac{1}{6}$ ft.-per-sec. per sec.).

Ans. $v = gt, \quad s = \frac{1}{2}gt^2;$

$v = 128\frac{2}{3}$ ft. per sec., $s = 257\frac{1}{3}$ ft.

(9) *A body falling in vacuo has a final velocity of* 250 *ft. per sec. Find the time of falling from rest, and the distance described* ($g = 32\frac{1}{6}$ ft.-per-sec. per sec.).

Ans. 7.77 sec.; 971.25 ft.

(10) *A body falling in vacuo from rest describes a distance of* 85 *feet. Find the time of fall and the final velocity* ($g = 32\frac{1}{6}$ ft.-per-sec. per sec.).

Ans. 2.3 sec.; 73.9 ft. per sec.

(11) *A body is projected vertically upwards in vacuo with a velocity of* 40 *ft. per sec. Find the height and the time of ascent* ($g = 32\frac{1}{6}$ ft.-per-sec. per sec.).

Ans. 24.87 ft.; 1.24 sec.

(12) *A body projected vertically upwards in vacuo returned after* $18\frac{1}{2}$ *seconds. Find the initial velocity and the height of ascent* ($g = 32\frac{1}{6}$ ft.-per-sec. per sec.).

Ans. 297.54 ft. per sec.; 1376 ft.

(13) *A body falling in vacuo has at a given instant a velocity of* 17 *ft. per sec., at a later instant a velocity of* 90 *ft. per sec. Find the time between the two instants and the distance traversed* ($g = 32\frac{1}{6}$ ft.-per-sec. per sec.).

Ans. 2.27 sec.; 121.44 ft.

(14) *A stone is dropped into a well and the splash is heard in* 3 *seconds. If sound travels in air with a uniform velocity of* 1090 *ft. per sec., find the depth of the well* ($g = 32\frac{1}{6}$ ft.-per-sec. per sec.).

Ans. 130.4 feet.

(15) *A point has two component velocities (or accelerations), at right angles, of* 35 *and* 87 *units. Find the resultant velocity (or accelerations).*

Ans. 93.77 units, making an angle with the 35 units of 68° 5′.

(16) *A point has a velocity of* 120 *ft. per sec. Resolve this into two component velocities at right angles,* (*a*) *one of the components being* 75 *ft. per sec.;* (*b*) *one of the components making an angle of* 34° 7′ 3″ *with the resultant.*

Ans. (*a*) 93.68 ft. per sec., the resultant making an angle with 75 ft. per sec. of 51° 19′ 4″.

(*b*) 99.343 ft. per sec. adjacent to the given angle and 67.306 ft. per sec. opposite.

(17) *A point has two accelerations of* 115 *and* 89 *ft.-per-sec. per sec. at an angle of* 147° 8′ 3″. (*a*) *Find the resultant acceleration;* (*b*) *Find the angle between the given accelerations when the resultant is equal to the lesser;* (*c*) *when it is equal to the greater.*

Ans. (*a*) 62.865 ft.-per-sec. per sec., making the angle of 83° 4′ with 89 ft.-per-sec. per sec.; (*b*) 130° 14′ 44″; (*c*) 112° 45′ 54″.

(18) *A point has an acceleration of* 77.5 *ft.-per-sec. per sec. Resolve it into two components* (*a*) *making with the given acceleration the angles* 35° 7′ 11″ *and* 52° 9′ 8″; (*b*) *when one of the components is* 50.5 *ft.-per-sec. per sec. and makes an angle of* 36° 8′ 6″ *with the re-*

sultant; (*c*) *when one of the components is* 60 *ft.-per-sec. per sec. and the other makes an angle of* 47° 10′ 11″ *with the resultant;* (*d*) *when the two components are* 46.2 *and* 35 *ft.-per-sec. per sec.*

Ans. (*a*) 61.265 and 44.634 ft.-per-sec. per sec.

(*b*) 47.27 ft.-per-sec. per sec., making an angle with the resultant of 39° 2′ 3″.

(*c*) 71.88 or 83.48 ft.-per-sec. per sec., making an angle with the resultant of 61° 28′ 17″ or 24° 9′ 4″.

(*d*) The components make an angle of 35° 4′, and the resultant makes with the component 46.2 the angle 15° 2′ 18″.

(19) *A stream flows with a velocity of* 1 *ft. per sec., and a boat whose speed is* 1.3 *ft. per sec. is steered up stream at an angle of* 30° *with the current. Find the resultant velocity.*

Ans. 0.66 ft. per sec. down stream at an angle of 79° 3′ with the current.

(20) *What is the ratio of the speed of light to that of a cannon-ball moving at the rate of* 2400 *feet per sec., if light passes from the sun to the earth, a distance of* 91,500,000 *miles, in* 8¼ *minutes?*

Ans. 402600 to 1.

(21) *ABC is a triangle. Two spheres of radii r_1 and r_2 start together from A and B, their centres moving along AC and BC with velocities which carry them separately to C in the same time. Find the distances each has gone through when they meet.*

Ans. If a, b and c are the sides of the triangle, the required distances are $\frac{b}{c}(c - r_1 - r_2)$, $\frac{a}{c}(c - r_1 - r_2)$.

(22) *If a particle is projected vertically in vacuo with a velocity of* $8g$, *find the time in which it will rise through the height* $14g$.

Ans. 2 sec. and 14 sec.

(23) *A body falling in vacuo is observed to describe* 144.9 *ft. and* 177.1 *ft. in two successive seconds. Find* g *and the time from the beginning of the motion.*

Ans. $g = 32.2$ ft.-per-sec. per sec.; $t = 4$ sec. to the beginning of the first of the two seconds.

(24) *A, B, C, D are points in a vertical line, the distances AB, BC, CD being equal. If a particle fall from A, prove that the times of describing AB, BC, CD are as*

$$1 : \sqrt{2} - 1 : \sqrt{3} - \sqrt{2}.$$

(25) *A particle describes in successive intervals of* 4 *seconds each spaces of* 24 *and* 64 *feet in the same straight line. Find the acceleration and the velocity at the beginning of the first interval.*

Ans. 2.5 ft.-per-sec. per sec.; one foot per sec.

(26) *A particle moves* 7 *ft. in the first second, and* 11 *and* 17 *ft. in the third and sixth seconds respectively. Show that these facts are consistent with the supposition of a uniform acceleration.*

(27) *A falling particle is observed at one portion of its path to pass through* n *ft. in* s *seconds. Find the distance described in the next* s *seconds.*

Ans. $n + gs^2$.

(28) *If* s, ms, *are the spaces described by a body in times* t, nt, *respectively, determine the acceleration and the velocity of projection.*

Ans. $\frac{2(m-n)s}{n(n-1)t^2}$ and $\frac{(m-n^2)s}{n(1-n)t}$.

(29) *If the focus of the path of a projectile be as much below the horizontal plane through the point of projection as the highest point of the path is above it, to find the angle of projection.*

Ans. If α_1 is the angle of projection, $\alpha_1 = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

(30) *Particles are projected from the same point in the same direction with different speeds. Find the locus of the foci of their paths.*

Ans. A straight line through the point of projection making an angle with the horizontal equal to $\frac{\pi}{2} - 2\alpha_1$, where α_1 is the angle of projection.

(31) *If a particle is projected in a direction inclined to the horizon, show that the time of moving between two points at the extremities of a focal chord of the parabolic path is proportional to the product of the velocities of the particle at the two points.*

(32) *Two particles are projected from two given points in the same vertical line in parallel directions and with equal speeds. Prove that tangents drawn to the path of the lower will cut off from the path of the upper arcs described in equal times.*

(33) *If a particle is projected from a given point so as to strike an inclined plane through that point at right angles, prove that* $\tan(\alpha_1 - \theta) = \frac{1}{2}\cot\theta$, *where* α_1 *is the angle which the direction of projection makes with the horizon, and* θ *is the inclination of the plane to the horizon.*

(34) *A particle is projected from a given point with a given velocity. Find the direction of projection in order that its path may touch a given plane.*

Ans. Let α_1 be the angle of projection with the horizontal and β the angle of the given plane with the horizontal, v_1 the velocity of projection and d the distance from the point of projection to where the given plane cuts the horizontal through the point of projection. Then

$$\cos(90 - \beta - \alpha_1) = \frac{\sqrt{gd \sin 2\beta}}{v_1}.$$

(35) *To find the least velocity with which a body can be projected from a given point so as to hit a given mark, and the direction of projection in this case.*

Ans. Let d be the horizontal distance from the point of projection to the mark, v_1 the velocity of projection, α_1 its inclination to the horizon and β the angle of elevation of the mark above the horizon. Then

$$v_1 = \sqrt{gd\frac{1+\sin\beta}{\cos\beta}}, \quad \alpha_1 = \frac{1}{4}\left(\pi + 2\beta\right).$$

(36) *If two circles the planes of which are vertical touch each other internally at their highest or lowest points, and if any chord be drawn within the larger circle, terminating respectively at its highest or lowest point, prove that the time of descent down that portion of the chord which is exterior to the smaller circle is invariable.*

(37) *AP, PB are chords of a circle, AB being the vertical diameter. Particles starting simultaneously from A, P, fall down AP, PB, respectively. Prove that the least distance between them is equal to the distance PB.*

(38) *Two circles lie in the same plane, the lowest point of one being in contact with the highest point of the other. Prove that the time of descent from any point of the former to a point of the latter, along a straight line joining these points and passing through the point of contact, is constant.*

(39) *A particle slides from rest down a smooth sloping roof and then falls to the ground. Find the point where it reaches the ground.*

Ans. Let $l =$ the length of the slope, α its inclination with the horizontal, $h =$ the height of the lowest point of the slope from the ground. Then the distance of the point where the particle reaches the ground from the foot of the wall is

$$2 \cos \alpha \sqrt{l \sin \alpha}\left(\sqrt{l \sin^3 \alpha + h} - \sqrt{l \sin^3 \alpha}\right).$$

(40) *Two equal inclined planes are placed back to back, and a particle projected up one flies over the top and comes to the ground just at the foot of the other. Find the velocity of projection, α being the inclination of each plane and h their common altitude.*

Ans. $\frac{1}{2}\sqrt{gh}\,(8 + \operatorname{cosec}^2 \alpha)^{\frac{1}{2}}$.

(41) *A particle is projected from a point A with the velocity acquired by falling down a height a, up an inclined plane of which the base and height are each equal to b, and after quitting the plane strikes the horizontal plane AB at the point B. Find AB.*

Ans. AB is equal to $a + (a^2 - b^2)^{\frac{1}{2}}$.

(43) *A particle slides down a smooth inclined plane. Determine the point at which the plane is cut by the directrix of the path described by the particle after leaving the plane.*

Ans. The directrix intersects the plane at the point where the particle began its motion.

(43) *A particle is projected up a rough plane, inclined to the horizon at an angle of 60°, with the velocity which it would have acquired in falling freely through a space of 12 ft., and just reaches the top of the plane. Find the altitude of the plane, its roughness being such that if it were inclined to the horizontal at an angle of 30°, the particle would be on the point of sliding.*

Ans. 9 ft.

(44) *A ring slides down a straight rod whilst the rod is carried uniformly in one plane, at a given angle to the horizon. Find the path described by the ring.*

Ans. A parabola.

(45) *A given circle and a given point are in the same vertical plane, the point being within the circle. Find the straight line of quickest descent from the point to the circle.*

Ans. The required straight line is the distance between the given point and the lower end of that chord of the circle which passes through the given point and terminates at the highest point of the circle.

(46) *A given point and a given straight line are in the same vertical plane. Determine the straight line of quickest descent from the given line to the point.*

Ans. From the given point P draw PX horizontally to meet the given line at X. Draw upwards along the line a length XY equal to PX. The straight line joining P and Y is the required straight line.

(47) *To determine the straight line of slowest descent from a given point to a given circle, the point being without the circle and both in the same vertical plane, the highest point of the circle being lower than the given point.*

Ans. Let P be the given point and Q the highest point of the circle. Join PQ and produce it to cut the circle at R. Then PR is the line required.

(48) *Determine the straight line of slowest descent from a given circle to a given point without it, the point and circle being in the same vertical plane, and the point being lower than the lowest point of the circle.*

Ans. From the given point draw an indefinite straight line cutting the circle at its lowest point, and the second intersection of the indefinite line and the circle is the required line.

(49) *Find the straight line of slowest descent from one given circle to another, both circles being in the same vertical plane, and each exterior to the other, the highest point of the latter circle being lower than the lowest of the former.*

Ans. Produce the line which joins the lowest point of the first circle and the highest point of the second, to meet both circles again. The distance between the second intersections is the required line.

(50) *A smooth tube of uniform bore is bent into the form of a circular arc greater than a semicircle, and placed in a vertical plane with its open ends upwards and in the same horizontal line. Find the velocity with which a ball that fits the tube must be projected along the interior from the lowest point, in order that it may pass out at one end and re-enter at the other.*

Ans. If r is the radius, h the depth of the centre of the circle below the horizontal through the two ends, v_1 the required velocity,

$$v_1^2 = \frac{g}{h}(r^2 + 2hr + 2h^2).$$

(51) *A particle slides from rest down a smooth tube in the form of the thread of a screw the axis of which is vertical. Find the time in which it will make a complete revolution about the axis.*

Ans. If r is the radius of the cylinder on which the helix is described, and α the angle which the thread makes with the generating line of the cylinder, the required time is

$$\left(\frac{8\pi r}{g \sin 2\alpha}\right)^{\frac{1}{2}}.$$

(52) *A particle falls to the lowest point of a cycloid down any arc of the curve, the axis of the cycloid being vertical and its vertex downwards. Prove that the vertical velocity is greatest when it has completed half its vertical descent.*

(53) *Also prove that it describes half the path in two thirds of the whole time.*

(54) *If a clock pendulum lose 5 sec. a day, determine the alteration which should be made in its length.*

Ans. It should be diminished by nearly the $\frac{1}{8640}$ part of its length.

(55) *A seconds pendulum was too long on a given day by a quantity a. It was then over-corrected so as to be too short by a during*

the next day. Prove that, l being the length of the seconds pendulum, the number of minutes gained in the two days was nearly

$$1080\frac{a^2}{l^2}.$$

(56) *A seconds pendulum carried to the top of a mountain is found to lose there 43.2 sec. a day. Find the height of the mountain, supposing the radius of the earth to be 4000 miles.*

Ans. 2 miles.

(57) *Find the time of vibration of a pendulum 20 feet long.*

Ans. Approximately 2.5 seconds.

(58) *A body dropped from the top of a wall falls to the ground while a pendulum 6 inches long makes 5 beats. Find the height of the wall.*

Ans. $\frac{25}{4}\pi^2$ feet.

(59) *A seconds pendulum is lengthened one hundredth of an inch. Find how many seconds it will lose daily.*

Ans. About 11 seconds.

(60) *If the length of a seconds pendulum is 39.1386 inches, what will be the length of one which vibrates 40 times a minute.*

Ans. 88.06185 inches.

(61) *If the length of a seconds pendulum is 39.1393 inches, find the value of g.*

Ans. $g = 32.190$ feet-per-sec. per sec.

(62) *A pendulum which beats seconds at the equator gains 5 minutes a day at the pole. Compare polar and equatorial gravity.*

Ans. 144 to 145, approximately.

(63) *Two pendulums the lengths of which are l_1 and l_2 vibrate at different points on the earth's surface. The number of vibrations which they make in the same time are in the ratio m_1 to m_2. Find the ratio of g at the two places.*

Ans. $\frac{l_1 m_1^2}{l_2 m_2^2}$.

(64) *A seconds pendulum is carried to the top of a mountain of which the height is 1 mile. Find the number of seconds it will lose daily, gravity being supposed to vary inversely as the square of the distance from the centre of the earth, and the radius of the earth to be 4000 miles.*

Ans. About 21.6 seconds.

(65) *A body revolves uniformly in a circle of 4 ft. radius in 10 seconds. Find the angular velocity and the velocity at the circumference.*

Ans. 0.628 radian per sec.; 2.5 ft. per sec.

(66) *A body revolves uniformly in a circle of 4 ft. radius with a velocity of 8 ft. per sec. Find the normal acceleration.*

Ans. 16 ft.-per-sec. per sec.

(67) *A body revolves uniformly in a circle of 18 ft. radius. The normal acceleration is 5 ft.-per-sec. per sec. Find the velocity.*

Ans. 9.5 ft. per sec.

(68) *A body whose velocity is* 10 *ft. per sec. is made to move uniformly in a circle by a normal acceleration of* 2 *ft.-per-sec. per sec. Find the radius.*

Ans. 50 ft.

(69) *Two points have velocities v_1 and v_2 and are made to move in a circle by reason of central accelerations inversely proportional to the square of the distance from the centre. The distance of one point is r_1. Find the distance r_2 of the other.*

Ans. $\frac{v_1^2}{r_1} : \frac{v_2^2}{r_2} = r_2^2 : r_1^2; \cdot r_2 = \left(\frac{v_1}{v_2}\right)^2 r_1.$

(70) *Find the relation between the distances r_1 and r_2 and the times of revolution t_1 and t_2.*

Ans. $$\frac{\left(\frac{2\pi r_1}{t_1}\right)^2}{r_1} : \frac{\left(\frac{2\pi r_2}{t_2}\right)^2}{r} = r_2^2 : r_1^2; \quad t_1^2 : t_2^2 = r_1^3 : r_2^3$$

KINEMATICS OF A RIGID SYSTEM.*

CHAPTER I.

RIGID SYSTEM WITH ONE POINT FIXED.

ROTATION. ANGULAR DISPLACEMENT. LINEAR DISPLACEMENT IN TERMS OF ANGULAR. LINE REPRESENTATIVE OF ANGULAR DISPLACEMENT. RESOLUTION AND COMPOSITION OF ANGULAR DISPLACEMENTS. ANGULAR VELOCITY. INSTANTANEOUS AXIS OF ROTATION. ANGULAR ACCELERATION. RESOLUTION AND COMPOSITION OF ANGULAR VELOCITIES AND ACCELERATIONS. EQUATIONS OF MOTION OF A POINT OF A ROTATING SYSTEM. MOMENT OF ANGULAR VELOCITY AND ACCELERATION. GENERAL ANALYTICAL DETERMINATION OF RESULTANT FOR CONCURRING ANGULAR VELOCITIES AND ACCELERATIONS.

Rotation.—When a rigid system moves so that all its points describe circles in parallel planes about a common straight line or *axis* passing through the centres of the circles and perpendicular to their planes, the system is said to **rotate** or have a **motion of rotation** about that axis. Any plane parallel to the planes of the circles is the **plane of rotation.**

Since the system is rigid, every point must describe its circle in the same time, or the angular speed (page 72) of every point is the same.

If the angular speed does not change and the plane of rotation does not change, the rotation of the system is **uniform.** If either the angular speed changes or the plane of rotation changes, the rotation is **variable.**

Motion of a Rigid System with One Point Fixed.—We have defined translation (page 13) as motion of a system such that every straight line joining every two points remains always parallel to itself during the motion. In such case the motion of the system is the same as that of any one of its points, and the study of the translation of a system is the same as the study of the translation of a point. In the preceding Chapters we have treated of the **kinematics of a point or translation.**

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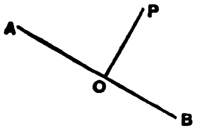
* The advanced student should read this portion of the work in connection with the analogous portions of *Statics* referred to in the text. The student new to the subject would do well to omit this portion of the work and take it in connection with Statics later.

If one point of a system is fixed, the motion of which it is capable will be more or less complex according as its points can or can not move relatively to each other. We restrict our discussion to *rigid systems*, that is, systems whose points can not move relatively to each other.

If one point of such a system is fixed, there can be no translation and the only motion of which it is capable is one of rotation as just defined.

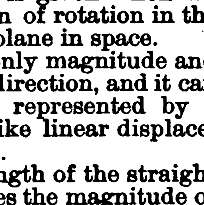
In such motion it is evident that all straight lines in the system must remain straight lines of unchanged length and mutual inclination, and all planes must remain planes of unchanged form, area and mutual inclination. Also the motions of any two points indefinitely near must be indefinitely nearly the same.



Angular Displacement of a Rigid System.—Let AB be the axis of rotation of a rigid system. Then, since every point must complete its circle in the same time, the angle described by all points in any given time must be the same as the angle described by any one point P.

The angle θ between the initial and final positions, in any given time, of the perpendicular PO from any point P to the axis, is called the **angular displacement** of the system.

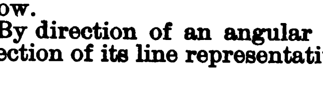
Since the angle θ is measured in radians, it is independent of the length PO (page 5).



Linear Displacement in Terms of Angular.—Let $OP_1 = OP_2 = r$ be the radius for any point P which moves in a circle perpendicular to the axis at O, through the angular displacement $P_1OP_2 = \theta$, from the initial position P_1 to the final position P_2. Then the triangle P_1OP_2 is isosceles, and if we draw ON perpendicular to P_1P_2 we have the linear displacement

$$P_1P_2 = 2r \sin \frac{\theta}{2}.$$

Line Representative of Angular Displacement of a Rigid System.—An angular displacement of a rigid system is given when we know not only its magnitude and the direction of rotation in the plane of rotation, but also the direction of that plane in space.



It is therefore a vector quantity having not only magnitude and sign, but also direction, and it can be completely represented by a straight line, like linear displacement (page 34).

Thus the length of the straight line AB denotes the magnitude of the angular displacement $P_1OP_2 = \theta$. The plane of rotation is at right angles to the line AB, *which is therefore coincident with the axis of rotation.* The direction of rotation is *always clockwise* when we look along this line in the direction indicated by the arrow.

By direction of an angular displacement we mean always the direction of its line representative as denoted by the arrow.

Composition and Resolution of Successive Angular Displacements.—Let a rigid system with one point fixed undergo successive angular displacements. It is required to determine the resultant angular displacement.

Evidently the successive angular displacements may be about the same or about different axes, and in either case may be finite or indefinitely small.

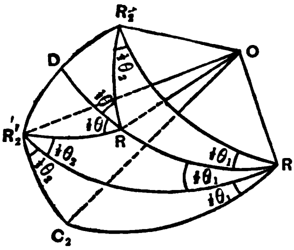
(*a*) **About the Same Axis—Finite or Indefinitely Small.**—If the axis of all the angular displacements is the same, the plane of rotation does not change, and the magnitude and sign or direction of the resultant displacement in that plane is given by the algebraic sum of the magnitudes of the successive displacements, whether they are finite or indefinitely small.

Inversely, any angular displacement about a given axis may be resolved into any number of successive displacements about the same axis, whether finite or indefinitely small, provided the algebraic sum of their magnitudes is equal to the magnitude of the given displacement and the same in sign.

(*b*) **About Different Axes—Displacements Finite.**—The axes must pass through the fixed point of the system. First let the displacements be finite.

Let O be the fixed point of the system, and OR_1, OR_2 the initial positions of the two given axes. Take $OR_1 = OR_2$, and let us suppose first a displacement θ_1 of the system about OR_1 and then a displacement θ_2 about the new position of the other axis. During this motion R_2 and R_1 will move on the surface of a sphere.

When the system is rotated an angle θ_1 about OR_1, the axis OR_2 moves from OR_2 to OR_2'.



Now join R_1R_2, R_1R_2' and R_2R_2' by great circles of the sphere. Then the angle $R_2R_1R_2' = \theta_1$.

Bisect this angle by a great circle meeting $R_2'R_2$ at D. Draw a great circle through R_2' inclined to $R_2'R_1$ at the angle $\frac{\theta_2}{2}$ and meeting R_1D at R.

Then draw $R_2'C_2$, making the same angle $\frac{\theta_2}{2}$ with $R_2'R_1$ on the other side, and make $R_2'C_2 = R_2'R$. Then R_1C_2 will equal R_1R, and the angle $R_2'R_1C_2 = \frac{\theta_1}{2}$.

When then the system is rotated about OR_1, and the axis OR_2 moves to OR_2' through the angle $R_2R_1R_2' = \theta_1$, the line OR will move to OC_2 through the angle $RR_1C_2 = \theta_1$.

If now the system is rotated about OR_2' through the angle $C_2R_2'R = \theta_2$, the line OC_2 moves back to OR.

Hence the line OR has the same position before and after the rotations. The resultant displacement is then a displacement about OR.

Hence, *the resultant of two successive rotations θ_1 about OR_1 and θ_2 about OR_2, when the axes intersect in a point O, is a single rotation θ about the axis OR passing through O.*

In order to find the position of this axis OR and the magnitude of θ, we have in the spherical triangle R_1RR_2 the angle $RR_1R_2 = \frac{1}{2}\theta_1$, the angle $RR_2R = \frac{1}{2}\theta_2$ and the exterior angle $DRR_2 = \frac{1}{2}\theta$. Hence

$$\cos\frac{1}{2}\theta = \cos\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2 - \sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2 \cos R_1R_2; \quad . \quad (1)$$

$$\frac{\sin R_1R}{\sin\frac{1}{2}\theta_2} = \frac{\sin RR_2}{\sin\frac{1}{2}\theta_1} = \frac{\sin R_1R_2}{\sin\frac{1}{2}\theta}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since OR_1 and OR_2 are lines of a rigid body and OR_1 coincides with the position of the first axis of rotation in space, the second axis of rotation in space has the position OR_2' and not OR_2. Hence, in general, *the order of the two successive rotations is not indifferent.*

Example.—*The telescope of a theodolite, originally horizontal and pointing north, is first turned into an altitude of* 60° *and then turned towards the west into the prime vertical. Find the resultant rotation.*

Ans. We have $\theta_1 = 60°$, $\theta_2 = 90°$, $R_1R_2 = 90°$. Hence

$$\cos\frac{1}{2}\theta = \frac{1}{2}\sqrt{3} \times \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{\frac{3}{2}}, \quad \text{or} \quad \sin\frac{1}{2}\theta = \frac{1}{2}\sqrt{\frac{5}{2}}.$$

For the position of the axis we have

$$\sin R_1R = \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}\sqrt{\frac{5}{2}}} = 2\sqrt{\frac{1}{5}}, \quad \sin RR_2 = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{\frac{5}{2}}} = \sqrt{\frac{2}{5}}.$$

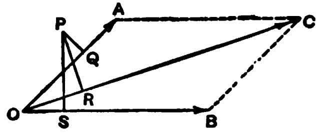
If we invert the order of the two rotations, we have $\theta_1 = 90°$, $\theta_2 = 60°$, $R_1R_2 = 90°$. Hence

$$\cos\frac{1}{2}\theta = \frac{1}{2}\sqrt{\frac{3}{2}}, \quad \text{or} \quad \sin\frac{1}{2}\theta = \frac{1}{2}\sqrt{\frac{5}{2}}, \text{ as before}$$

For the position of the axis

$$\sin R_1R = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{\frac{5}{2}}} = \sqrt{\frac{2}{5}}, \quad \sin RR_2 = \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}\sqrt{\frac{5}{2}}} = 2\sqrt{\frac{1}{5}}.$$

(c) About Different Axes—The Displacements Indefinitely Small.—



Second let the rotations be indefinitely small. Let $OA = \theta$ and $OB = \phi$ be the line representatives. Complete the parallelogram and draw OC. Let P be any point of the system in the plane of OA and OB, and draw the perpendiculars PQ, PR, PS. When rotation occurs about OA, the point P will move perpendicularly to the plane of the paper through a very small distance represented by $r\theta$ or $OA \times PQ$ (page 5).

When rotation occurs about OB, the point P moves perpendicularly to the plane of the paper also, through a very small distance represented by $OB \times PS$. Since both these displacements are very small they coincide in direction, and the resultant is

$$OA \times PQ + OB \times PS = OC \times PR.$$

Hence the resultant displacement is given by OC.

We should have the same result if the rotation about *OB* occurred first, also if the point *P* had been taken within the angle *AOB*. Also whether *OA* and *OB* are axes fixed in the body or in space.

If we have more than two successive rotations, the third may be compounded with the resultant of the first two in like manner.

Hence if a rigid system with one point fixed undergo any number of successive indefinitely small angular displacements about different axes either fixed in the system or fixed in space, the resultant angular displacement is obtained by treating the line representatives precisely like linear displacements (page 35). We have thus the parallelogram and polygon of angular displacements.

Composition and Resolution of Simultaneous Angular Displacements.—The simultaneous angular displacements may be finite or indefinitely small and must be either about the same axis or different axes.

(*a*) **About the Same Axis.**—If the axis of all the angular displacements is the same, the plane of rotation does not change and the magnitude and sign or direction of the resultant displacement in that plane is given by the algebraic sum of the magnitudes of the simultaneous angular displacements, whether they are finite or indefinitely small.

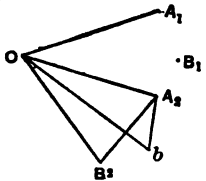
(*b*) **About Different Axes.**—If the rotations *OA*, *OB* are indefinitely small, we see from the figure, page 171, that it makes no difference whether they are successive or simultaneous. We can resolve and combine them, therefore, by their line representatives just like linear displacements (page 35). We have then the parallelogram and polygon of angular displacements.

If the rotations *OA*, *OB* are finite, we can divide each up into a number of indefinitely small rotations and treat each pair as before. We have then the parallelogram and polygon of angular displacements in this case also.

Composition and Resolution of Angular Displacements in general.—We see then that *in all cases except finite successive angular displacements about different axes* we can combine and resolve any number of angular displacements whether simultaneous or successive, finite or indefinitely small, about the same or about different axes by means of the line representatives, just like linear displacements.

Simultaneous angular displacements are usually called *component angular displacements.* Component angular displacements must then be understood to always mean simultaneous angular displacements, unless otherwise specified.

Sign of Components of Angular Displacement.—The sign of the line representatives of the components along the axes *X*, *Y*, *Z* of an angular displacement follows the same rule as for linear displacement (page 36). Hence if we look along the line representatives *towards the origin*, the radius vector *will always be seen to move counter-clockwise.*



Axis of Rotation.—In every possible displacement of a rigid system with one point fixed, there is one line fixed in the system passing through the fixed point, called the **axis of rotation**, which has the same position in both the initial and final positions of the system.

Let *O* be the fixed point of the system, and let A_1, B_1 be the initial and A_2, B_2 the final positions in space of two points of the system.

Since the system is rigid, $OA_1 = OA_2$, and $OB_1 = OB_2$. Let A_1 be brought to A_2 by rotation about an axis through O perpendicular to the plane of A_1OA_2. By this rotation B_1 moves to b, and since the system is rigid, $Ob = OB_1 = OB_2$, and $A_2b = A_2B_2$. The triangles OA_2b and OA_2B_2 are then equal in all respects, and b can be brought to B_2 by rotation about OA_2. The given displacement can always then be produced by two successive rotations about two axes passing through O. As we have seen (page 171), two such successive rotations give as resultant a single rotation about an axis through O. This is the **axis of rotation.**

COR. 1. Hence any angular displacement of a rigid system with one point fixed is completely specified by the line representative of the resultant angular displacement, which coincides in direction with the axis of rotation.

COR. 2. Any angular displacement of a rigid system with one point fixed may be resolved into three angular displacements about the co-ordinate axes through the fixed point taken as origin.

COR. 3. Every line in the system parallel to the axis of rotation remains unchanged in direction.

Mean Angular Velocity of a Rigid System.—The magnitude of the angular displacement during a given time of a rigid system with one point fixed, divided by the number of units of time, gives the magnitude of the **mean angular velocity** of the system.

It is represented by a line just like angular displacement (page 170). By direction of mean angular velocity we always mean direction of the line representative.

Mean angular speed then is mean time-rate of angle described (page 72). Mean angular velocity is mean time-rate of angular displacement.

Instantaneous Angular Velocity of a Rigid System.—The limiting magnitude and direction of the mean angular velocity when the interval of time is indefinitely small is the **instantaneous angular velocity.**

The term angular velocity always signifies instantaneous angular velocity unless otherwise specified.

It may be represented by a straight line just like angular displacement (page 170).

By direction of an angular velocity we always mean the direction of its line representative.

We see then that angular displacement and angular velocity are vector quantities like linear displacement and linear velocity. Angular velocity is directed angular speed, just as linear velocity is directed linear speed. Speed is magnitude of velocity, whether linear or angular (page 43).

Instantaneous Axis of Rotation. — The instantaneous angular velocity of a rigid system is then given by its line representative. This line representative coincides in position with the axis of rotation at the instant. This axis is then the **instantaneous axis of rotation.**

Unit of Angular Velocity.—Since the magnitude of the angular velocity at any instant is the angular speed in a given direction at that instant, the unit of angular velocity is the same as for angular speed, or one radian per sec. We denote the magnitude then by the same letter, ω, and we have the same numeric equations as for angular speed (page 73).

Thus for mean angular velocity

$$\omega = \frac{\theta - \theta_1}{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for instantaneous angular velocity

$$\omega = \frac{d\theta}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

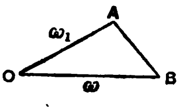
Uniform and Variable Angular Velocity.—Angular velocity is *uniform* when the line representative has the same magnitude and direction whatever the interval of time. Uniform angular velocity is then uniform angular speed in an unchanging plane, just as uniform linear velocity is uniform linear speed in an unchanging direction (page 43).

In such case angular velocity is the same as the mean angular velocity for any interval of time.

When either the magnitude or direction of the angular velocity changes it is *variable*.

When the magnitude alone changes we have variable angular speed in an unchanged plane of rotation. When the direction only changes we have uniform angular speed in a changing plane of rotation. When both change we have variable angular speed in a changing plane of rotation.

Mean Angular Acceleration of a Rigid System.—If $OA = \omega_1$ and $OB = \omega$ are the line representatives of the initial and final angular velocities of a rigid system with one point fixed, during any time t, then AB is the line representative of the **integral angular acceleration** of the system during the time t, and $\frac{AB}{t}$ gives the magnitude of the **mean angular acceleration** whose direction is AB. (Compare page 48.)



Mean angular acceleration then is time-rate of change of angular velocity, whether that change takes place in the direction of the angular velocity or not.

Instantaneous Angular Acceleration of a Rigid System. —The limiting magnitude and direction of the mean angular acceleration when the interval of time is indefinitely small is the **instantaneous** angular acceleration. It is the limiting time-rate of change of angular velocity whether that change takes place in the direction of the angular velocity or not.

Angular acceleration always signifies instantaneous angular acceleration unless otherwise specified.

It may be represented by a straight line just like angular displacement (page 170). By direction of an angular acceleration we mean the direction of its line representative.

Instantaneous Axis of Angular Acceleration.—The instantaneous angular acceleration of a rigid system is then given by its line representative. This line representative coincides in position with the axis of angular acceleration at the instant. This axis is then the **instantaneous axis of angular acceleration.**

Angular acceleration may be zero, uniform or variable. When it is zero, the angular velocity is uniform and we have uniform angular speed and an unchanging plane of rotation.

When it is uniform, it has the same magnitude and the same direction whatever the interval of time. In such case the acceleration is equal to the mean acceleration for any interval of time. If the direction coincides with that of the initial velocity, we have uniform rate of change of angular speed and an unchanged plane of rotation. If it makes an angle with the velocity, we have a changing plane of rotation and variable velocity.

When it is variable, either direction or magnitude changes or both change.

If the angular acceleration is always at right angles to the angular velocity, it only changes the direction but not the magnitude of that velocity.

Hence, just as on page 53 a normal linear acceleration has no effect upon the linear speed, but only changes the direction of motion, so, if a rigid system rotating with given angular speed about an axis has an angular acceleration about an axis always perpendicular to the first, there is change of direction of this axis but no change of angular speed about it.

The gyroscope is an illustration of this principle.

Resolution and Composition of Angular Velocity and Acceleration.—Since for an indefinitely small time the angular displacement is indefinitely small, we see from page 171 that we can combine angular velocities and accelerations, whether simultaneous or successive, by means of their line representatives just like linear velocities and accelerations (page 43).

Sign of Components of Angular Velocity and Acceleration.—The sign of the line representatives of the components along the axes X, Y, Z of an angular velocity or acceleration follows the same rule as for linear velocities and accelerations (pages 44, 50).

Unit of Angular Acceleration.—Angular acceleration is measured in terms of the same unit as rate of change of angular speed (page 73), or one radian-per-sec. per sec. We denote its magnitude then by the same letter, α.

Relations between Angular and Linear Velocity and Acceleration.—We have also the same relations between angular and linear acceleration and velocity as for a point moving in a circle (page 76).

Thus we have, for any point of a rigid system whose distance from the axis of rotation at any distance is r,

$$r\omega = v, \quad r\alpha = f_t, \quad v\omega = f_n = r\omega^2 = \frac{v^2}{r}, \quad fp = f_t r = r^2\alpha, \quad vr = r^2\omega.$$

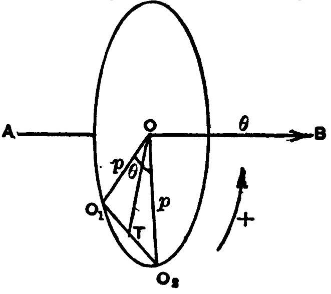
Equations of Motion of a Rotating Rigid System under Different Angular Accelerations.—Since angular velocities and accelerations are represented by straight lines, just like linear velocities and accelerations, we have the same equations for motion of a rotating rigid system as on page 50. We have only to substitute ω for v, θ for s, α for f.

With these substitutions equations (1) to (14), page 50, hold good and it is unnecessary to repeat them here.

Moment of Angular Displacement.—Just as we called the product of the magnitude of a linear displacement by the magnitude of the perpendicular let fall from any given point upon its direction the *moment* of the linear displacement (page 60), so for angular displacement we call the product of its magnitude by the magnitude of the perpendicular from any point upon the direction of the line representative the *moment* of the angular displacement.

We take its sign just as for moment of linear displacement, page 62. Since the line representative is coincident with the axis, the perpendicular is the distance of the point from the axis.

Thus if $AB = \theta$ is the line representative of an angular displacement $O_1OO_2 = \theta$ of a rigid system, the axis has the position



AOB. If then O_1 is the initial position of any point of the system and $OO_1 = p$ is the perpendicular from O_1 upon the axis or direction of the line representative *AB*, the moment is $\pm p\theta$ according to direction, just as for moment of linear displacement (page 62). But $p\theta$ is the *length of the arc* O_1O_2 described by the point O_1 in a plane perpendicular to the plane of *AB* and OO_1.

Hence, *the moment $p\theta$ of the angular displacement θ of a rigid system relative to any point of the system gives the length of the arc O_1O_2 described by that point in a plane perpendicular to the plane of the axis AB and the radius vector p.*

The corresponding linear displacement of O_1 is evidently

$$T = 2p \sin \frac{\theta}{2} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Since the angle OO_1O_2 equals the angle OO_2O_1, we have for the direction of the linear displacement relative to OO_1,

$$\text{angle } OO_1O_2 = 90° - O_1OT = \frac{\pi - \theta}{2} \cdot \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

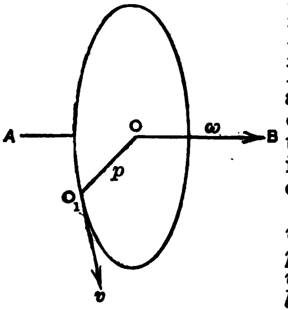
We have also, just as on page 62, the algebraic sum of the moments of any number of component angular displacements, relative to any point, equal to the moment of the resultant.

Also, just as on page 60, the line representative of an angular displacement may be laid off from any point in its line of direction without affecting its moment.

Moment of Angular Velocity or Acceleration. — Just as we called the product of the magnitude of a linear velocity or acceleration by the magnitude of the perpendicular from any given point upon its direction the *moment* of the linear velocity or acceleration, so for angular velocity or acceleration we call the product of the magnitude by the magnitude of the perpendicular from any point upon the direction of the line representative the *moment* of the angular velocity or acceleration.

We take its sign just as for moment of linear velocity or acceleration (page 60). Since the line representative is coincident with the axis, the perpendicular is the distance of the point from the axis.

Thus if $AB = \omega$ is the line representative of an angular velocity of a rigid system, the instantaneous axis has the position *AOB*. If then O_1 is any point of the system and $OO_1 = p$ is the perpendicular from O_1 on the axis or direction of the line representative, the moment is $\pm p\omega$ according to direction, just as for moment of linear velocity (page 60). But this is the linear velocity v of O_1 at the instant, in a direction perpendicular to the plane of *AB* and OO_1.



Hence, *the moment $p\omega$ of the angular velocity ω of a rigid system relative to any point of the system gives the linear velocity v of that point in a direction perpendicular to the plane of the instantaneous axis of rotation AB and the instantaneous radius vector p.*

In the same way, *the moment $p\alpha$ of the angular acceleration α of a rigid system relative to any point of the system gives the linear tangential acceleration f_t of that point in a direction perpendicular to the plane of the instantaneous axis of angular acceleration and the instantaneous radius vector.*

We have also, just as on page 62, the algebraic sum of the moments of any number of component angular velocities or accelerations, relative to any point, equal to the moment of the resultant.

Also, just as on page 60, the line representative of an angular velocity or acceleration may be laid off from any point in its line of direction without affecting its moment.

Concurring Angular Displacements, Velocities or Accelerations.—We see then that angular displacements, velocities or accelerations are represented by straight lines, called *line representatives*, which coincide with the axis of rotation. We deal with them entirely by means of these line representatives. When we speak of their "direction," we mean the direction of the line representatives. We resolve and combine them by means of their line representatives, and in the same way we have their moments just as for linear displacements, velocities or accelerations. Following the same analogy, we can speak of them as "applied" or "acting" at certain points. When they all intersect at the same point, we may call them **concurring**, just as if they were linear. When they do not intersect at the same point they are **non-concurring**. When they act in the same direction in the same line they are **conspiring**. When in the same or opposite directions in parallel lines they are **parallel**. When in opposite directions in the same line or in parallel lines they are **opposite**. When they lie in the same plane they are **co-planar**.

Condition for Rotation only.—If a rigid system has one point fixed, it can have no translation but only rotation, and therefore all the component angular displacements, velocities or accelerations must reduce to a concurring system, so that we have a single resultant angular displacement, velocity or acceleration about an axis through this point, which is therefore at rest.

General Analytical Determination of Resultant Angular Displacement, Velocity or Acceleration for any Number of Concurring Components.—We see then that all the equations of pages 63 to 65 hold good for angular displacements, velocities or accelerations, as well as for linear.

For angular displacements we have only to substitute θ in place of v. The moments M_x, M_y, M_z then give the *arcs of displacement* about the axes of X, Y, Z of the origin, considered as a point of the rigid system, rotating about the resultant axis.

For angular velocities we have only to substitute ω for v. The moments M_x, M_y, M_z then give the component linear velocities v_x, v_y, v_z along the axes of X, Y, Z *of the origin*, considered as a point of the rigid system, rotating about the resultant axis.

For angular accelerations we have only to substitute α for v. The moments M_x, M_y, M_z then give the component linear tangential accelerations f_{tx}, f_{ty}, f_{tz} along the axes of X, Y, Z *of the origin*, considered as a point of the rigid system, rotating about the resultant axis.

To make our notation consistent we should also replace $\cos a$, $\cos b$, $\cos c$, page 65, by $\cos d$, $\cos e$, $\cos f$, and replace $\cos d$, $\cos e$, $\cos f$, page 66, by $\cos a$, $\cos b$, $\cos c$.

We have then from page 65, equation (4), for the component linear velocities v_x, v_y, v_z along the axes of X, Y, Z *of the origin*, considered as a point of the rigid system, rotating about the resultant axis,

$$\left.\begin{aligned} v_x &= \omega_z y - \omega_y z; \\ v_y &= \omega_x z - \omega_z x; \\ v_z &= \omega_y x - \omega_x y. \end{aligned}\right\} \quad \dots\dots\dots \quad (1)$$

We have also in the same way

$$\left.\begin{aligned} f_{tx} &= \alpha_z y - \alpha_y z; \\ f_{ty} &= \alpha_x z - \alpha_z x; \\ f_{tz} &= \alpha_y x - \alpha_x y. \end{aligned}\right\} \quad \dots\dots\dots \quad (2)$$

Equations (2) give the component linear tangential accelerations along the axes of X, Y, Z of the origin, considered as a point of the rigid system, rotating about the resultant axis.

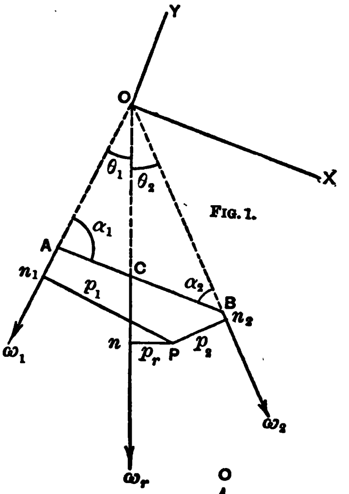
If we multiply the first of equations (1) by ω_x, the second by ω_y, the third by ω_z and add, we obtain

$$v_x\omega_x + v_y\omega_y + v_z\omega_z = 0. \quad \dots\dots\dots \quad (3)$$

Equation (3) is the *condition for rotation only*. When it is fulfilled, we know that the motion of the system is that of rotation only about the instantaneous axis.

Resultant of Two Concurring Component Angular Displacements, Velocities or Accelerations.*—It will be of profit to specially discuss the case of two concurring component angular displacements, velocities or accelerations.

Let the two angular velocities ω_1, ω_2 be in the same plane and pass through the points A and B of a rigid system, so that they intersect at O. Then the resultant ω_r must pass through O and be in the plane of ω_1, ω_2.

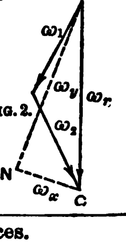


Take any point P in this plane and draw the perpendiculars $Pn_1 = p_1$, $Pn_2 = p_2$, $Pn = p_r$. Then, since the moment of the resultant is equal to the algebraic sum of the moments of the components,

$$\omega_r p_r = \omega_1 p_1 + \omega_2 p_2, \quad . \quad (1)$$

where regard must be paid to the signs in any case. Thus we have in the figure

$$\omega_r p_r = \omega_1 p_1 - \omega_2 p_2.$$



Draw the line AB, intersecting the resultant ω_r at the point C. Let α_1 be the angle of ω_1, and α_2 the angle of ω_2 with AB. If we take moments about C, we have

$$\omega_1 . AC \sin\alpha_1 = \omega_2 . BC \sin\alpha_2.$$

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* Compare Statics—Non-concurring Forces.

But $AC + BC = AB$. Hence

$$AC = \frac{\omega_2 \,.\, AB \sin \alpha_2}{\omega_1 \sin \alpha_1 + \omega_2 \sin \alpha_2}, \quad BC = \frac{\omega_1 \,.\, AB \sin \alpha_1}{\omega_1 \sin \alpha_1 + \omega_2 \sin \alpha_2}. \quad . (2)$$

We thus know the *position* of the resultant ω_r in the plane of ω_1 and ω_2.

Magnitude and Direction of the Resultant.*—If we lay off in Fig. 2, ω_1 and ω_2, then, just as for linear velocities, $OC = \omega_r$ gives the magnitude and direction of the resultant.

Take rectangular axes OX, OY, Fig. 1, in the plane of ω_1, ω_2, and let OX be parallel to AB. Let ω_1 make the angle α_1 with OX and β_1 with OY, and ω_2 make the angle α_2 with OX and β_2 with OY. Denote the algebraic sum of the components parallel to OX by ω_x and parallel to OY by ω_y. Then we have

$$\left.\begin{aligned} \omega_x &= \omega_1 \cos \alpha_1 + \omega_2 \cos \alpha_2; \\ \omega_y &= \omega_1 \cos \beta_1 + \omega_2 \cos \beta_2; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

where we must pay regard to signs. Thus components in the direction OX, OY are positive, in the opposite directions negative.

If the resultant ω_r makes the angles d and e with the axes of X and Y, we have

$$\cos d = \frac{\omega_x}{\omega_r} \quad \cos e = \frac{\omega_y}{\omega_r}. \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

Squaring and adding,

$$\omega_r = \sqrt{\omega_x^2 + \omega_y^2}. \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

The magnitude and direction of the resultant are thus determined.

Also if θ_1 is the angle of ω_1 with the resultant, and θ_2 the angle of ω_2 with the resultant, and θ the angle between ω_1 and ω_2, we have directly from Fig. 2

$$\sin \theta_1 = \frac{\omega_2}{\omega_r} \sin \theta, \quad \sin \theta_2 = \frac{\omega_1}{\omega_r} \sin \theta, \; . \; . \; . \; . \; . \quad (6)$$

and

$$\omega_r = \sqrt{\omega_1^2 + \omega_2^2 \pm 2\omega_1\omega_2 \cos \theta}, \; . \; . \; . \; . \; . \quad (7)$$

where the (+) sign is used when θ is less than 90°, and the (—) sign when θ is greater than 90°.

The tangent of the angle d which the resultant makes with AB or OX is

$$\tan d = \frac{\omega_y}{\omega_x}. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

From (6) and (7) we can find the magnitude and direction of the resultant directly if θ is known. If α_1 and α_2 are known, (3) and (5) give ω_r, and (4) or (8) the direction.

From (1) we have also

$$p_r = \frac{\omega_1 p_1 + \omega_2 p_2}{\omega_r}, \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

where regard must be had for the signs of $\omega_1 p_1$ and $\omega_2 p_2$ in any case. From (9) for any given point P for which p_1 and p_2 are known, we can locate the resultant by describing a circle with centre P and radius p_r and drawing ω_r tangent to this circle in the direction given by (6).

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* Compare Statics.

The same formulas hold good for two concurring angular accelerations. We have only to replace ω by α. [The student will of course not confuse this α, which stands for angular acceleration, with α_1, α_2 in the formulas, which stand for angles.]

The same formulas hold good also for two concurring component angular displacements. We have only to replace ω by d.

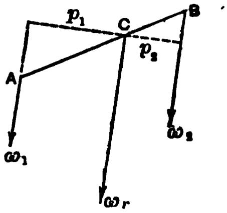
When the Angular Displacements, Velocities or Accelerations are Parallel.—In this case α_1 and α_2 are equal, $\theta = 0$, the intersection O is at an infinite distance, $\omega_r = \omega_1 + \omega_2$, and we have from (2)

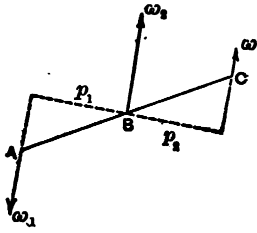
$$AC = \frac{\omega_2}{\omega_r} \cdot AB, \quad BC = \frac{\omega_1}{\omega_r} \cdot AB; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and hence, multiplying the first by ω_1 and the second by ω_2,

$$\omega_1 \cdot AC = \omega_2 \cdot BC, \quad \text{or} \quad \frac{\omega_1}{\omega_2} = \frac{BC}{AC}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To prove this independently, take C as the point of moments.





Then whether the line representatives act in the same or in opposite directions, we have

$$\omega_1 p_1 - \omega_2 p_2 = 0, \quad \text{or} \quad \omega_1 p_1 = \omega_2 p_2.$$

But from similar triangles

$$\frac{p_1}{p_2} = \frac{AC}{BC}, \quad \text{hence} \quad \frac{\omega_1}{\omega_2} = \frac{BC}{AC}.$$

The same holds for angular displacements or accelerations.

We see from (1) that the distances AC and BC depend only upon the magnitudes of ω_1 and ω_2 and the distance AB, and not at all upon the common direction of ω_1 and ω_2. Therefore if ω_1 and ω_2 always pass through the points A and B no matter what their common direction, the resultant ω_r always passes through C. The point C is then the point of application of the resultant ω_r for all directions.

Hence, *the resultant of two parallel component angular displacements, velocities or accelerations is in their plane and equal in magnitude to their algebraic sum. It acts parallel to the components in the direction of the greater. If the components always pass through two given points A and B, the resultant always passes through a point C no matter what the common direction. This point C is then the point of application of the resultant. It is on the straight line AB or this line produced, and divides it into segments inversely as the components. Or the products of the components into their adjacent segments are equal.* (Compare Statics—Parallel Forces.)

Cor. 1. When the components act in the same direction, the resultant lies within the components and nearest the larger. When the components act in opposite directions, the resultant lies without the components and on the side of the larger.

Cor. 2. When the components are opposite and equal in magnitude, $\omega_r = 0$. Also from (1), $AC = \infty$, $BC = \infty$, or *the resultant is zero and acts at an infinite distance.*

That is, equal and opposite parallel components cannot have a single resultant.

Such a system is called a **couple.** (Compare Statics—Parallel Forces.)

EXAMPLES.

(1) *A rigid system has two component rotations of* 2 *and* 4 *radians about axes inclined* 60°. *Find the resultant rotation.*

Ans. Component rotations are understood to be simultaneous unless otherwise specified (page 172). Hence magnitude of resultant rotation is $2\sqrt{7}$ radians; axis inclined at an angle with the greater component whose sine is $\frac{\sqrt{3}}{2\sqrt{7}}$.

(2) *A sphere with one of its superficial points fixed has two component rotations—one of* 8 *radians about a tangent line and one of* 15 *radians about a diameter. Find the axis of the resultant displacement and the number of complete revolutions made about it.*

Ans. Inclination of axis to greater component at an angle whose tang is $\frac{8}{15}$. Resultant displacement 17 radians, number of complete revolutions $\frac{17}{2\pi}$.

(3) *A sphere is rotating uniformly about a diameter at the rate of* 10 *radians per min. Find* (*a*) *the component angular velocity about another diameter inclined* 30° *to the former, and* (*b*) *the component rotation produced in* 2 *min. about a diameter inclined* 45° *to the first.*

Ans. (*a*) $5\sqrt{3}$ radians per min.; (*b*) $10\sqrt{2}$ radians.

(4) *A pendulum suspended at a point in the polar axis of the earth oscillates in a vertical plane. Find the motion of this plane relative to the earth.*

Ans. The plane of the pendulum is fixed in space, and the motion of the earth with reference to this plane is a rotation from west to east at the rate of one revolution per day. The motion of the plane relative to the earth is then from *east to west* at the same rate of one revolution per day.

(5) *A pendulum is hung at a place of latitude* λ *and oscillates in a vertical plane. Find* (*a*) *the angular velocity of the plane of the pendulum's motion relative to the earth, and* (*b*) *the time in which this plane will make one complete revolution at a place in latitude* 60° *N.*

Ans. The angular velocity of the earth about its axis is 2π radians per day. The component of this in the direction of an axis through the centre of the earth and the point of suspension of the pendulum is $2\pi \sin \lambda$ radians per day from west to east. This is the motion of the earth relative to the plane of the pendulum. Hence—

(*a*) The motion of the plane of the pendulum relative to the earth is $2\pi \sin \lambda$ radians per day from east to west;

(*b*) The time of revolution is $\dfrac{2\pi}{2\pi \sin \lambda} = \dfrac{1}{\sin 60} = \dfrac{2}{\sqrt{3}}$ days.

(6) *A cube rotates about a vertically-upward axis through one of its edges. At a given instant at which the diagonal of the upper surface passing through the axis points north the cube has an angular velocity of* 40 *radians per sec., and begins to have a uniform angular acceleration about an axis vertically downwards through the same edge of* 6 *rad.-per-sec. per sec. Find* (*a*) *the direction in which the diagonal will point after* 20 *sec.*; (*b*) *the number of revolutions made by the cube.*

Ans. From the equations of motion page 73 we have $\omega = \omega_1 - \alpha t$, $\theta - \theta_1 = \omega_1 t - \dfrac{1}{2}\alpha t^2$. We have $\omega_1 = 40$ radians per sec., $\alpha = 6$ rad.-per-sec. per sec., $t = 20$ sec., $\theta_1 = 0$.

In the time $t_1 = \dfrac{\omega_1}{\alpha} = \dfrac{20}{3}$ sec. the cube comes to rest and has the angular displacement $\theta_1 = \dfrac{400}{3}$ radians towards the east.

It then moves in the opposite direction towards the west during the time $t = 20 - \dfrac{20}{3} = \dfrac{40}{3}$ sec. and undergoes the angular displacement $\theta = \dfrac{1}{2}\alpha t^2 = \dfrac{1600}{3}$ radians. Hence (*a*) the angular displacement from the north point towards the west is 400 radians or 63.661 revolutions, or 63 revolutions and 237°.96 W. The direction of the diagonal is then S. 57°.96 E. (*b*) The total angular displacement is $\dfrac{2000}{3}$ radians, hence the number of revolutions is $\dfrac{333.3}{\pi}$.

(7) *A sphere is rotating at a given instant about a given diameter ACB with an angular velocity of* 4 *rad. per min. It has an angular acceleration of* 2 *rad.-per-min. per min. about a diameter inclined* 30° *to ACB. Find* (*a*) *the angular velocity, and* (*b*) *the angular displacement after* 20 *min.*

Ans. (Page 174.) (*a*) $4\sqrt{101 + 10\sqrt{3}}$ rad. per min. inclined to CB at an angle whose tangent is $\dfrac{5}{1 + 5\sqrt{3}}$; (*b*) $80\sqrt{26 + 5\sqrt{3}}$ radians inclined to CB at any angle whose tangent is $\dfrac{5}{2 + 5\sqrt{3}}$.

(8) *A rigid system has one point fixed. The co-ordinates of this point with reference to any point of the system taken as origin are at any instant* $x = +3$ *ft.,* $y = +4$ *ft.,* $z = 0$. *The component angular velocities at this instant are* $\omega_1 = 40$, $\omega_2 = 50$, $\omega_3 = 60$ *radians per sec., the line representatives making the angles* $\alpha_1 = 60°$, $\beta_1 = 150°$, $\gamma_1 = 90°$; $\alpha_2 = 120°$, $\beta_2 = 30°$, $\gamma_2 = 90°$; $\alpha_3 = 120°$, $\beta_3 = 150°$, $\gamma_3 = 90°$. *Find the resultant angular velocity.*

Ans. (See Example (1), page 67.) The component angular velocities are in one plane and

$$\omega_x = -35 \text{ radians per sec.}, \quad \omega_y = -43.3 \text{ radians per sec.}$$

The resultant is $\omega_r = 55.67$ radians per sec., its line representative or the instantaneous axis of rotation making with the horizontal the angle $d = 128° \, 57' \, 17''$, and with the vertical the angle $e = 141° \, 2' \, 43''$. If we look along this line representative which passes through the fixed point, towards the origin, the rotation will be seen as counter-clockwise.

The moment of the resultant angular velocity ω_r with reference to the point O gives us the linear velocity of rotation at O about the instantaneous axis, $v_r = +10$ ft. per sec. in a direction through O perpendicular to the plane XY, or along Z, from O towards Z.

The distance of O from the axis is $p =$ about 0.18 ft.

The equation of the axis is $y = 1.237x + 0.286$. Its intercepts on the axis are $y' = +0.286$ ft., $x' = -0.232$ ft.

(9) *Express and solve the same example for component angular accelerations and displacements.*

(10) *A rigid system has one point fixed. The co-ordinates of this point with reference to any point of the system taken as an origin are at any instant $x = +3$ ft., $y = +4$ ft., $z = +5$ ft. The component angular velocities at this instant are $\omega_1 = 40$, $\omega_2 = 50$, $\omega_3 = 60$ radians per sec., the line representatives making the angles with the axes $\alpha_1 = 60°$, $\beta_1 = 100°$, γ_1 obtuse; $\alpha_2 = 100°$, $\beta_2 = 60°$, γ_2 acute; $\alpha_3 = 120°$, $\beta_3 = 100°$, γ_3 acute. Find the resultant angular velocity.*

Ans. (See Example (2), page 67.) We have

$$\omega_x = -18.6824, \quad \omega_y = +7.635, \quad \omega_z = +59.391 \text{ radians per sec.}$$

The resultant angular velocity is $\omega_r = 62.73$ radians per sec., its line representative making with the axes the angles

$$d = 118° \, 17' \, 33'', \quad e = 85° \, 6' \, 12'', \quad f = 12° \, 30' \, 24''.$$

This line representative passes through the fixed point and gives the instantaneous axis of rotation. If we look along this line towards the origin, the rotation will be seen as counter-clockwise.

The velocities of rotation at O along the axes are

$$v_z = +97.6346, \quad v_x = +199.389, \quad v_y = -271.585 \text{ ft. per sec.}$$

The resultant velocity of rotation at O is $v_r = 407.6$ ft. per sec., making with the axes angles

$$a = 60° \, 42' \, 57'', \quad b = 131° \, 46' \, 24'', \quad c = 76° \, 8' \, 31''.$$

The equations of the projections of the axis upon the co-ordinate planes are:

on plane XY, $y = -0.408x + 5.226$;
" " YZ, $z = +7.778y - 26.115$;
" " ZX, $x = -0.314z + 4.572$.

The axis pierces the plane XY at $x' = +4.572$ ft., $y' = +3.357$ ft.;
" " " " " YZ " $y' = +5.226$ ft., $z' = -14.56$ ft.;
" " " " " ZX " $z' = -26.115$ ft., $x' = +12.809$ ft.

(11) *Express and solve the same example for component angular accelerations and displacements.*

(12) *Let the axes of two concurring angular velocities of a rigid system, $\omega_1 = 20$, $\omega_2 = 30$ radians per sec., pass through the points A, B of the system, the distance $AB = 2$ ft., and the angles $\alpha_1 = 60°$, $\alpha_2 = 30°$. Find the point C on the line AB through which the resultant axis passes, and the magnitude and direction of the resultant angular velocity.*

Ans. $AC = 0.928$ ft., $\omega_x = +15.98$ radians per sec., $\omega_y = -32.32$ radians per sec. The angle of the resultant with AB is given by

$$\tan d = -\frac{32.32}{15.98} = -2.022, \quad \text{or} \quad d = 63^\circ\ 41' = BC\omega_r.$$

The resultant is

$$\omega_r = 36.05 \text{ radians per sec.}$$

We have also for the angle of the resultant with ω_1, since $\theta = 90^\circ$,

$$\sin\theta_1 = \frac{30}{36.05} = 0.832, \quad \text{or} \quad \theta_1 = 56^\circ\ 19'.$$

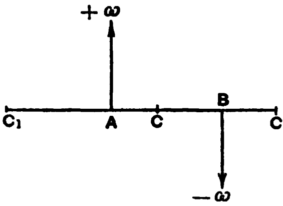
(13) *Express and solve the same example for component angular accelerations and displacements.*

CHAPTER II.

MOMENT OF A COUPLE.

DISPLACEMENT OF A RIGID SYSTEM. RIGID PLANE SYSTEM. COMPOSITION AND RESOLUTION OF TRANSLATION AND ANGULAR DISPLACEMENT. COMPOSITION AND RESOLUTION OF TRANSLATION AND ANGULAR VELOCITY. CENTRAL AXIS. SCREW MOTION. ROTATION AND RECTILINEAR TRANSLATION. COMBINED PARALLEL ROTATIONS, ONE AXIS FIXED. INTERSECTING AXES, ONE AXIS FIXED. ANALYTIC DETERMINATION OF RESULTANT ANGULAR VELOCITY AND VELOCITY OF TRANSLATION FOR NON-CONCURRING ANGULAR VELOCITIES.

Moment of a Couple.*—We have just seen in the preceding Chapter, page 181, that two parallel equal and opposite components acting at different points of a rigid system constitute a *couple.* We may have then an angular-displacement couple or angular-velocity couple or angular-acceleration couple.



Let $+\omega$, $-\omega$, acting at the points A, B of a rigid system, constitute an angular-velocity couple.

If we take any point C between the components, or any point C_1, C_2 on either side, in the plane of the components, we have in the first case, denoting the distance AB by p, for the moment about C, just as for linear velocities (page 60),

$$-\omega . AC - \omega . BC = -\omega(AC + BC) = -p\omega.$$

In the second case, for the moment about C_1 we have

$$\omega . C_1A - \omega . C_1B = -\omega(C_1B - C_1A) = -p\omega.$$

In the third case, for the moment about C_2 we have

$$-\omega . C_2A + \omega . C_2B = -\omega(C_2A - C_2B) = -p\omega.$$

Hence the moment about every point in the plane of the couple is constant and equal to $\pm p\omega$, the (+) or (—) sign denoting direction just as for moment of linear velocity (page 60).

For an angular-acceleration couple we have in the same way $\pm p\alpha$, for an angular-displacement couple $\pm p\theta$.

We see then *that the moment of a couple is the same for every point in its plane and equal to the product of either of the components by the distance between them.*

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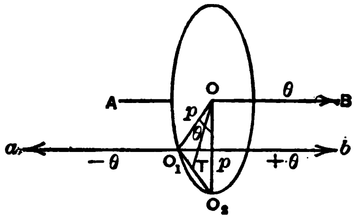
* Compare Statics—Parallel Forces.

Composition and Resolution of Translation and Angular Displacement.—Let a rigid system have a rotation of $O_1OO_2 = \theta$ radians about an axis AOB through the point O, and $AB = \theta$ be the line representative.



If we take any other point of the system, as O_1, and at this point apply the two equal and opposite angular displacements $O_1a = -\theta$ and $O_1b = +\theta$, both parallel to AB, it is evident that the motion of the system is not affected. We have then the angular displacement about the axis AOB reduced to an equal angular displacement O_1b about a parallel axis through O_1 *and a couple* AB and O_1a.

The moment of this couple is the same for every point in its plane and equal to $p\theta$, where p is the perpendicular distance between the components AB and O_1a of the couple.

But we have seen (page 177) that the moment $p\theta$ corresponds to a linear displacement in a plane perpendicular to the plane of the couple of

$$T = 2p \sin \frac{\theta}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

making an angle OO_1O_2 with p given by

$$OO_1O_2 = \frac{\pi - \theta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, *an angular displacement* θ *about any given axis can be resolved into an equal angular displacement about a parallel axis through any point of the system and a linear translation in the plane of rotation of the system whose magnitude and direction are given by* (1) *and* (2).

Conversely, *the resultant of the rotation of a rigid system about a given axis and a translation in any given direction, is an equal rotation about a parallel axis, whose position with reference to the first can be determined by* (1) *and* (2).

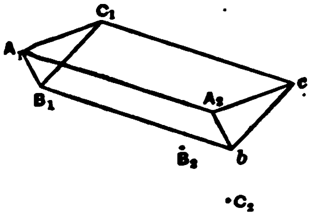
COR. 1. Two non-concurring angular displacements can be reduced to a resultant angular displacement about a resultant axis at any point and a couple which causes translation.

COR. 2. Hence if we have any number of component angular rotations about any axes, whether these axes intersect or not, we can reduce each to an equal rotation about a parallel axis through some one point of the system and a translation of the system.

The resultant translation can then be found as on page 35, and the resultant rotation as on page 173, for simultaneous angular displacements.

COR. 3. Therefore any number of component translations and rotations can all be reduced to a single translation and a single rotation about any given point. It is evident that this single rotation is not affected by the position of the point, which affects the translation only.

Displacement of a Rigid System.—Any displacement of a rigid system may be produced by a translation and an angular displacement.



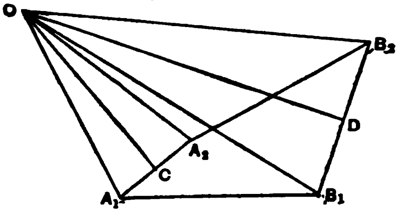
Let A_1, B_1, C_1 be the positions of any three points which determine the initial position of the rigid system. Let A_2, B_2, C_2 be the final position of these points after any displacement. First let the system be translated, so that A_1 comes to its final position A_2. Then B_1 and C_1 will take the positions b and c, the lines B_1b and C_1c being equal and parallel to A_1A_2. We see then that A_2 is a fixed point in the system so far as the two positions $A_1B_1C_1$, $A_2B_2C_2$ are concerned.

But we have seen (page 173) that in every possible displacement of a rigid system with one point fixed there is an axis of rotation fixed in the system which remains unchanged. Hence A_2cb can be brought to the position $A_2B_2C_2$ by rotation about that axis.

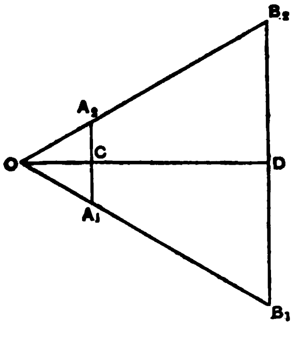
COR. 1. It follows that the displacement of a rigid system is known if the magnitude and direction of the linear displacement of any point is known, and also the magnitude and direction of the angular displacement of the system about that point.

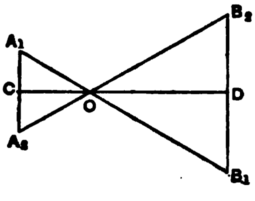
COR. 2. Also, the displacement of a rigid system is known if the magnitude and direction of the component linear displacements of any point parallel to three rectangular axes and of the component angular displacements of the system about axes parallel to the first through the point are known.

Rigid Plane System.—Any displacement of a rigid plane system in its own plane may be produced by rotation about some point in the plane.



Let A_1B_1 and A_2B_2 be the initial and final positions in the plane, of the same line of the system, so that A_1B_1 and A_2B_2 are of equal length. Join A_1A_2 and B_1B_2 by lines and bisect these lines at C and D. Erect perpendiculars at the points of bisection C and D and produce them to intersection at O. Then by construction $OA_1 = OA_2$, and $OB_1 = OB_2$, and $A_1B_1 = A_2B_2$. Hence the two triangles OA_1B_1 and OA_2B_2 are in all respects equal and the line A_1B_1 may be brought to coincide with A_2B_2 by rotation about the point O.





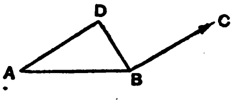
If A_2B_2 is parallel to A_1B_1, we have translation only and the point O is at an infinite distance.

Since the angle A_1OB_1 = the angle A_2OB_2, if we take the angle A_2OB_1 from both we have $A_1OA_2 = B_1OB_2$. If then the displacement is such that A_1 falls on OB_1 or on OB_1 produced, A_2 must be on OB_2 or OB_2 produced.

In both cases OC and OD coincide and do not intersect, but it is evident that in such case the point O in which A_1B_1 and A_2B_2 intersect is the point about which rotation would produce the given displacement.

If in any case A_1A_2 and B_1B_2 are indefinitely small, the point O is called the **instantaneous centre** of motion.

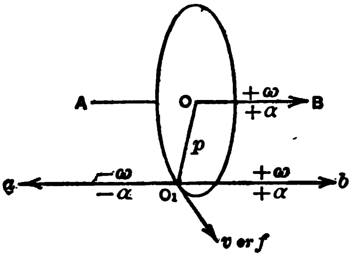
Any Displacement of a Rigid System.—Any displacement of a rigid system may be produced by rotation about an axis and a translation in the direction of that axis.



Let AB and BC represent the resultant translation and rotation to which the component translations and rotations of the system can be reduced (page 187, Cor. 3).

Draw AD and DB parallel and perpendicular to BC. Then the translation AB is resolved into the two components AD and DB. But the resultant of DB and BC (page 187) is an equal rotation about an axis parallel to BC. Hence the translation AD and the rotation BC are reducible to the translation AD and a rotation about an axis parallel to AD.

Composition and Resolution of Translation and Angular Velocity or Angular Acceleration. — Let a rigid system have an $\left\{\begin{matrix}\text{angular velocity}\\ \text{angular acceleration}\end{matrix}\right\}$ of $\left\{\begin{matrix}\omega \text{ radians per sec.}\\ \alpha \text{ radians-per-sec. per sec.}\end{matrix}\right\}$ about an axis AOB through the point O, and let $AB = \left\{\begin{matrix}+\omega\\ +\alpha\end{matrix}\right\}$ be the linear representative.



If we take any other point of the system, as O_1, and at this point apply two equal and opposite $\left\{\begin{matrix}\text{angular velocities}\\ \text{angular accelerations}\end{matrix}\right\}$, $O_1a = \left\{\begin{matrix}-\omega\\ -\alpha\end{matrix}\right\}$ and $O_1b = \left\{\begin{matrix}+\omega\\ +\alpha\end{matrix}\right\}$, it is evident that the motion of the system is not affected. We have then the $\left\{\begin{matrix}\text{angular velocity}\\ \text{angular acceleration}\end{matrix}\right\}$ about the axis AOB reduced to an equal $\left\{\begin{matrix}\text{angular velocity}\\ \text{angular acceleration}\end{matrix}\right\}$ O_1b about a parallel axis through O_1 and a couple represented by AB and O_1a.

The moment of this couple is the same for every point in its plane (page 186) and equal to $\left\{\begin{matrix}p\omega\\ p\alpha\end{matrix}\right\}$, where p is the perpendicular distance between the two axes.

But we have seen (page 177) that the moment $\left\{\begin{matrix}p\omega\\ p\alpha\end{matrix}\right\}$ gives the linear $\left\{\begin{matrix}\text{velocity } v\\ \text{acceleration } f\end{matrix}\right\}$ in a direction perpendicular to the plane of the couple. Since the moment of the couple is the same for every point in its plane, we have then translation of the entire system in a direction perpendicular to the plane of the couple, as well as simultaneous rotation about the axis through O_1. The direction and magnitude of this translation will depend upon the point O_1, but the rotation will be the same wherever the point O_1 may be taken.

Hence,* *an* $\left\{\begin{matrix}\textit{angular velocity}\\ \textit{angular acceleration}\end{matrix}\right\}$ *of a rigid system about any axis can be resolved into an equal* $\left\{\begin{matrix}\textit{angular velocity}\\ \textit{angular acceleration}\end{matrix}\right\}$ *about a parallel axis at any distance* p, *and a* $\left\{\begin{matrix}\textit{velocity}\\ \textit{acceleration}\end{matrix}\right\}$ *of translation* $\left\{\begin{matrix}v = p\omega\\ f = p\alpha\end{matrix}\right\}$ *in a direction at right angles to the plane of the axes.*

Conversely, *the resultant of an* $\left\{\begin{matrix}\textit{angular velocity } \omega\\ \textit{angular acceleration } \alpha\end{matrix}\right\}$ *of a rigid system about a given axis and a* $\left\{\begin{matrix}\textit{velocity}\\ \textit{acceleration}\end{matrix}\right\}$ *of translation* $\left\{\begin{matrix}v\\ f\end{matrix}\right\}$ *in any direction is an equal* $\left\{\begin{matrix}\textit{angular velocity}\\ \textit{angular acceleration}\end{matrix}\right\}$ *about a parallel axis distant* $\left\{\begin{matrix}p = \dfrac{v}{\omega}\\ p = \dfrac{f}{\alpha}\end{matrix}\right\}$ *in a direction perpendicular to the plane of* $\left\{\begin{matrix}v\\ f\end{matrix}\right\}$ *and the given axis.*

This parallel axis is the **instantaneous axis.**

COR. 1. Hence if we have any number of component $\left\{\begin{matrix}\text{angular velocities}\\ \text{angular accelerations}\end{matrix}\right\}$ about any axis, each can be reduced to an equal $\left\{\begin{matrix}\text{angular velocity}\\ \text{angular acceleration}\end{matrix}\right\}$ about a parallel axis through any one point of the system, and a $\left\{\begin{matrix}\text{velocity}\\ \text{acceleration}\end{matrix}\right\}$ of translation of the system. We can then find the resultant $\left\{\begin{matrix}\text{velocity}\\ \text{acceleration}\end{matrix}\right\}$ of translation as on page $\left\{\begin{matrix}43\\ 49\end{matrix}\right\}$ and the resultant $\left\{\begin{matrix}\text{angular velocity}\\ \text{angular acceleration}\end{matrix}\right\}$ as on page 176.

COR. 2. Therefore any number of component $\left\{\begin{matrix}\text{angular velocities}\\ \text{angular accelerations}\end{matrix}\right\}$ and $\left\{\begin{matrix}\text{velocities}\\ \text{accelerations}\end{matrix}\right\}$ of translation, can all be reduced to a single resultant $\left\{\begin{matrix}\text{velocity}\\ \text{acceleration}\end{matrix}\right\}$ of translation and a single resultant $\left\{\begin{matrix}\text{angular velocity}\\ \text{angular translation}\end{matrix}\right\}$ about an axis through any one point of the system. The $\left\{\begin{matrix}\text{velocity}\\ \text{acceleration}\end{matrix}\right\}$ of translation will vary in direction and magnitude with the point chosen. The $\left\{\begin{matrix}\text{angular velocity}\\ \text{angular acceleration}\end{matrix}\right\}$ will be the same no matter what point is chosen.

Central Axis.—Any number of component angular velocities of a rigid system can be reduced to a single angular velocity about a determinate axis and a simultaneous velocity of translation of the system along that axis. Such an axis is called the **central axis**, and such motion is called **screw motion.**

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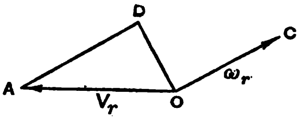
* Compare Statics—Non-concurring Forces.

Thus let *OA* and *OC* represent the resultant linear velocity of translation V_r and the resultant angular velocity ω_r, to which, as we have just seen, all the angular velocities can be reduced.

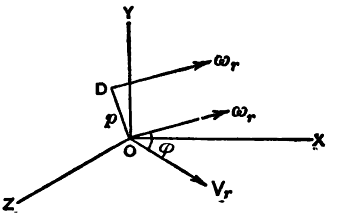


Draw *AD* and *OD* parallel and perpendicular to *OC*. Then the velocity of translation $OA = V_r$ is resolved into the two components *AD* and *OD*. But the resultant of *OC* and *OD* is an equal angular velocity about an axis parallel to *OC* (page 190).

Hence the velocity of translation $OA = V_r$ and the angular velocity $OC = \omega_r$ are reducible to an equal angular velocity about an axis parallel to *OC* and a linear velocity of translation *AD* along that axis.

This axis is called the *central axis*, and may be located by the following geometric construction.

At any point *O* of the system taken arbitrarily let the velocity of translation be V_r and the rotation axis through *O* be ω_r, making the angle ϕ. Through *O* draw a line $OD = p$ perpendicular to the plane of V_r and ω_r, so that $p\omega_r = V_r \sin \phi$, or $p = \dfrac{V_r \sin \phi}{\omega_r}$.



Then a line through *D* parallel to the rotation axis at *O* will be the central axis. (Compare Statics—Non-concurring Forces.

Screw Motion.—Let u_r denote the resultant velocity of translation along the central axis. This is called the **velocity of advance.** The distance *d* advanced during one complete rotation of the system is called the **pitch of the screw,** and the distance advanced during a rotation of one radian, or $\frac{d}{2\pi}$, we call the **unit pitch** of the screw.

If ω_r is the resultant angular velocity of rotation, the time of a complete rotation is $t = \frac{2\pi}{\omega_r}$.

We have then for the value of the pitch

$$d = u_r t = \frac{2\pi u_r}{\omega_r}, \quad \text{hence} \quad u_r = \frac{\omega_r d}{2\pi}, \quad . \quad . \quad . \quad . \quad (1)$$

and for the unit pitch

$$\frac{d}{2\pi} = \frac{u_r}{\omega_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If *r* is the radius vector of any point of the system, then the linear velocity of that point due to rotation about the axis is

$$v = r\omega_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in a direction perpendicular to the plane of the central axis and the radius vector.

The resultant velocity at that point is then

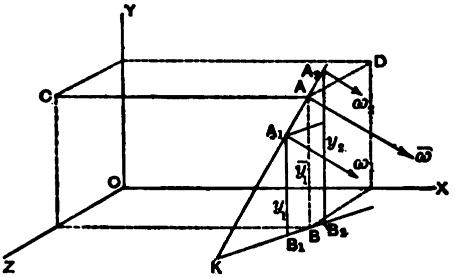
$$v_r = \sqrt{u_r^2 + v^2} = \omega_r\sqrt{\frac{d^2}{4\pi^2} + r^2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The inclination of the path at that point to the plane of rotation is given by

$$\tan i = \frac{u_r}{v} = \frac{d}{2\pi r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or the tangent of the angle of inclination at any point is equal to the ratio of the pitch to the circumference of the circle described by the point relatively to the axis; or it is equal to the ratio of the unit pitch to the radius vector of the point.

Centre of Parallel Angular Velocities.*—Let ω_1, ω_2, ω_3, etc., be any number of parallel angular velocities passing through the points A_1, A_2, A_3, etc., of a rigid system.



Then the resultant ω_r must be parallel to the components and equal in magnitude to their algebraic sum, or

$$\omega_r = \omega_1 + \omega_2 + \omega_3 + \ldots = \Sigma\omega.$$

Take any two components ω_1 and ω_2, and produce the line A_1, A_2 to intersection K with the plane ZX. Drop perpendiculars A_1B_1, A_2B_2 to this plane and draw the line KB_1B_2 in this plane.

Now, from page 181, the resultant of ω_1 and ω_2 is $\bar{\omega}_1 = \omega_1 + \omega_2$ and its point of application is at A on the line A_1A_2, so that

$$\frac{\omega_1}{\omega_2} = \frac{A_2A}{A_1A}.$$

Drop the perpendicular AB to the plane ZX. Then we have by similar triangles

$$\frac{A_2A}{A_1A} = \frac{B_2B}{B_1B}.$$

Denote the distance A_1B_1, A_2B_2 by y_1, y_2, respectively, and the distance AB, or the ordinate of the point of application of the resultant $\bar{\omega}_1$ of ω_1 and ω_2, by $\bar{y}_1$. Then we have by similar triangles

$$\frac{B_2B}{B_1B} = \frac{y_2 - \bar{y}_1}{\bar{y}_1 - y_1}.$$

Hence

$$\frac{\omega_1}{\omega_2} = \frac{y_2 - \bar{y}_1}{\bar{y}_1 - y_1}, \quad \text{or} \quad \bar{y}_1 = \frac{\omega_1 y_1 + \omega_2 y_2}{\omega_1 + \omega_2}.$$

In the same way for three angular velocities, ω_1, ω_2, ω_3, we can combine the resultant $\bar{\omega}_1$ of ω_1 and ω_2 passing through A, with ω_3.

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* Compare Statics—Parallel Forces.

We thus obtain for the ordinate of the point of application of the resultant of three forces

$$\bar{y}_3 = \frac{\omega_1 y_1 + \omega_2 y_2 + \omega_3 y_3}{\omega_1 + \omega_2 + \omega_3}.$$

In general, then, for any number of parallel angular velocities we have for the ordinate $\bar{y}$ of the point of application of the resultant

$$\bar{y} = \frac{\Sigma \omega y}{\Sigma \omega}. \qquad \qquad (1)$$

In precisely similar manner, if we denote the distances AC and AD of the point of application of the resultant from the planes YZ and XY by $\bar{x}$ and $\bar{y}$, we have

$$\bar{x} = \frac{\Sigma \omega x}{\Sigma \omega}; \qquad \qquad (2)$$

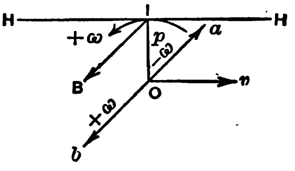
$$\bar{z} = \frac{\Sigma \omega z}{\Sigma \omega}. \qquad \qquad (3)$$

Equations (1), (2) and (3) give the co-ordinates of the point of application of the resultant for any number of parallel angular velocities. This point is called the *centre of parallel angular velocities*. The same equations evidently hold for parallel angular accelerations, by replacing ω by α. We have then the *centre of parallel angular accelerations*.

The position of this centre depends only upon the magnitude and position of the line representatives and is independent of their common direction.

If $\bar{z}$ is zero, z_1, z_2, etc., are zero, and the line representatives all lie in the plane XY. The centre is then given by (1) and (2). If $\bar{z}$ and $\bar{y}$ are zero, the centre is in the axis of X and is given by (2).

Rotation and Rectilinear Translation Combined.—Let a rigid system have an angular velocity $Ob = \omega$ about an axis through O, perpendicular to the plane of the paper, and at the same time a velocity of translation v in a straight line. Then, as we have seen, page 177, v can be replaced by the couple Oa and IB, and we have at any instant a resultant rotation $IB = \omega$ about a parallel axis though I at a distance $OI = p = \frac{v}{\omega}$ in a direction perpendicular to that of v. This axis is the instantaneous axis; that is, the point I at any distant has the velocity v in one direction due to translation, and the velocity $v = p\omega$ in the other direction due to the couple, and is therefore at rest.



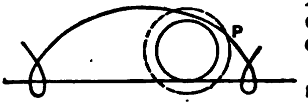
It is evident that every straight line in the system parallel to the moving axis at O and at a constant distance from O of $p = \frac{v}{\omega}$ becomes in turn the instantaneous axis when it arrives at the position I with reference to O.

Hence when a rigid system has a velocity of translation in a straight line and at the same time an angular velocity ω about a given axis Ob, the resultant motion of the system *is the same as if* *a cylindrical surface fixed in the system, of radius* $p = \frac{v}{\omega}$, *rolled on a plane HIH parallel to the plane of Ob and v.*

The path described by any point in the axis *Ob* is a straight line. The path described by any point not in this axis is called a **trochoid.** The special form of trochoid described by any point in the cylindrical surface is called a **cycloid.** Any internal point describes a **prolate cycloid**; any external point, a **curtate cycloid.**

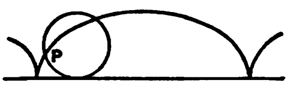
The general form of these curves is shown in the accompanying figures.

Curtate Cycloid

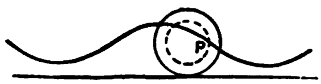


A common illustration of such motion is a wheel rolling in a straight line on a plane. If the radius is p, we have $p\omega = v$ and hence $\frac{v}{\omega} = p$. The instantaneous axis is at right angles to the plane of the wheel and passes through the point of contact with the plane. The velocity at the centre is v, and at the opposite extremity of the diameter through the point of contact $2v$ in the direction of translation. The velocity of any point at a distance d from the instantaneous axis is $d\omega$ in a direction perpendicular to the plane of the instantaneous axis and the instantaneous radius vector d.

Cycloid



Prolate Cycloid



Combined Parallel Rotations—One Axis Fixed.—Let a rigid system rotate with the angular velocity ω_1 about a moving axis at O_1, and at the same time let this axis revolve with the angular velocity ω_2 about a parallel fixed axis at O_2.

Then, as we have seen, page 181, the resultant ω_r is in the plane of the components ω_1 and ω_2, is equal in magnitude to their algebraic sum and divides the straight line joining O_1 and O_2 into segments inversely as the components. Also when the components act in the same direction the resultant lies within the components, and when in opposite directions without the components and on the side of the larger.

Fig. 1, then, represents the case in which ω_1 and ω_2 are in the same direction; Fig. 2, that in which ω_1 and ω_2 are in opposite directions and ω_1 is the greater; Fig. 3, that in which ω_1 and ω_2 are in opposite directions and ω_2 is the greater.

FIG. 1.

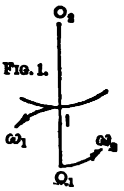


FIG. 2.

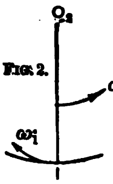
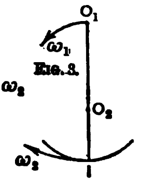


FIG. 3.



The resultant angular velocity is in all cases then given by

$$\omega_r = \omega_1 + \omega_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where we take ω_1 and ω_2 with their proper signs.

This resultant rotation ω_r takes place about the instantaneous axis through I, so situated that (page 181)

$$\frac{\omega_1}{\omega_2} = \frac{IO_2}{IO_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that at any instant I has two opposite and equal linear velocities and is therefore at that instant at rest. Since then $\omega_1 . IO_1 = \omega_2 . IO_2$, we see at once that in Fig. 2, ω_1 is greater than ω_2, and in Fig. 3, ω_1 is less than ω_2.

We have also as on page 181, taking moments about O_2 and O_1,

$$\left.\begin{aligned}\omega_1 . O_1O_2 &= \omega_r . IO_2, \quad\text{or}\quad IO_2 = \frac{\omega_1}{\omega_r} . O_1O_2;\\ \omega_2 . O_1O_2 &= \omega_r . IO_1, \quad\text{or}\quad IO_1 = \frac{\omega_2}{\omega_r} . O_1O_2.\end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (3)$$

All the lines in the system which successively occupy the position of the instantaneous axis are then situated in a cylindrical surface described about O_1 with the radius $IO_1 = \frac{\omega_2}{\omega_r} . O_1O_2$; and all the positions of the instantaneous axis are contained in a cylindrical surface described about O_2 with the radius $IO_2 = \frac{\omega_1}{\omega_r} . O_1O_2$.

Hence the resultant motion of a rigid system which rotates about an axis O_1 while at the same time this axis revolves about a fixed axis O_2 *is the same as if a cylindrical surface of radius* $IO_1 = \frac{\omega_2}{\omega_r} . O_1O_2$, *fixed in the system, rolls upon a fixed cylindrical surface of radius* $IO_2 = \frac{\omega_1}{\omega_r} . O_1O_2$.

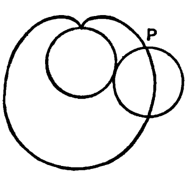
In Fig. 1, a convex cylinder rolls on a convex cylinder; in Fig. 2, a smaller convex cylinder rolls within a larger concave cylinder; in Fig. 3, a larger concave cylinder rolls upon a smaller convex cylinder.

The path described by any point in the moving axis through O_1 relatively to the fixed axis at C_1 is a circle.

The path described by any point relatively to the fixed axis is called an **epitrochoid** when the rolling cylinder is outside of the fixed cylinder and an **hypotrochoid** when it is inside. The special form of epitrochoid or hypotrochoid described by a point in the surface of the rolling cylinder is called an **epicycloid** when the rolling cylinder is outside and an **hypocycloid** when it is inside the fixed cylinder.

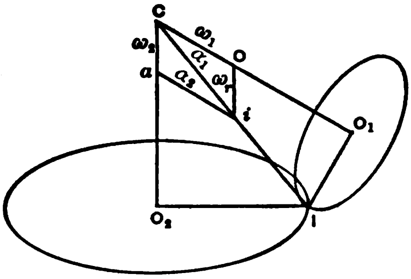
When the distance O_1O_2 is infinite we have the case of the preceding Article, of a cylinder rolling on a straight line. In this case $\omega_2 = 0$.

When a cylinder rolls externally upon another of equal size, the special form of epicycloid described by a point in its surface is called the **cardioid**. In this case ω_1 and ω are equal and in the same direction.



When a cylinder, as in Fig. 2, rolls within a concave cylinder of double its radius, we have $\omega_1 = 2\omega_2$. In this case each point in the surface of the rolling cylinder moves to and fro in a straight line, being a diameter of the fixed cylinder; each point in the axis of the rolling cylinder describes a circle of the same radius as that cylinder, and any other point in or without the rolling cylinder describes an ellipse of greater or less eccentricity, having its centre in the fixed axis at C. This principle has been made available in instruments for drawing and turning ellipses.

Rotation about Intersecting Axes—One Axis Fixed.—Let CO_2 be a fixed axis and about it let the plane O_2CO_1 rotate with the angular velocity ω_2. Let CO_1 be an axis in the rotating plane, and about that axis let a rigid system rotate with the angular velocity ω_1 relatively to the rotating plane.



If we lay off from C the line representatives $Ca = \omega_2$ and $CO = \omega_1$ along the axes, the diagonal Ci of the parallelogram gives the magnitude and direction of the resultant angular velocity ω_r. The instantaneous axis then occupies the position CI. If we denote the angles ICO_2 and ICO_1 by α_2 and α_1, we have

$$\tan \alpha_2 = \frac{\omega_1 \sin (\alpha_1 + \alpha_2)}{\omega_2 + \omega_1 \cos (\alpha_1 + \alpha_2)} = \frac{\frac{\omega_1}{\omega_2} \sin (\alpha_1 + \alpha_2)}{1 + \frac{\omega_1}{\omega_2} \cos (\alpha_1 + \alpha_2)}; \qquad (1)$$

$$\tan \alpha_1 = \frac{\sin (\alpha_1 + \alpha_2)}{\frac{\omega_1}{\omega_2} + \cos (\alpha_1 + \alpha_2)}. \qquad (2)$$

From (1) and (2) we can find α_1 and α_2, when the angle between the axes $(\alpha_1 + \alpha_2)$ and the angular velocity ratio $\frac{\omega_1}{\omega_2}$ are given. We have also

$$\omega_r^2 = \omega_1^2 + \omega_2^2 \pm 2\omega_1\omega_2 \cos (\alpha_1 + \alpha_2), \qquad (3)$$

and

$$\frac{\omega_r}{\omega_2} = \frac{\sin (\alpha_1 + \alpha_2)}{\sin \alpha_1}, \quad \frac{\omega_r}{\omega_1} = \frac{\sin (\alpha_1 + \alpha_2)}{\sin \alpha_2}, \quad \frac{\omega_1}{\omega_2} = \frac{\sin \alpha_2}{\sin \alpha_1}. \qquad (4)$$

All lines which come successively into the position of the instantaneous axis are in the surface of a cone described by the revolution of CI about CO_1; and all the positions of the instantaneous axis lie in the surface of a cone described by the revolution of CI about CO_2.

Hence the motion of the rigid system *is such as would be produced by the rolling of the cone CIO_1, fixed in the system, about the fixed cone CIO_2.*

If r_2 is the radius O_2I of the fixed cone, and r_1 the radius O_1I of the rolling cone, we have

$$r_1\omega_1 = r_2\omega_2, \text{ or } \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}. \qquad (5)$$

For the height $CO_2 = h$ of the fixed cone we have

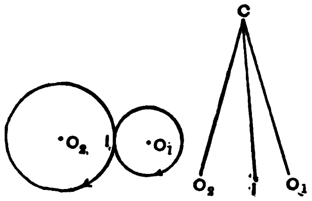
$$h_2 = r_2 \operatorname{cotang} \alpha_2 = \frac{r_1 + r_2 \cos (\alpha_1 + \alpha_2)}{\sin (\alpha_1 + \alpha_2)}, \qquad (6)$$

and for the height $CO_1 = h_1$ of the rolling cone

$$h_1 = r_1 \text{ cotang } \alpha_1 = \frac{r_2 + r_1 \cos(\alpha_1 + \alpha_2)}{\sin(\alpha_1 + \alpha_2)}. \quad . \quad . \quad (7)$$

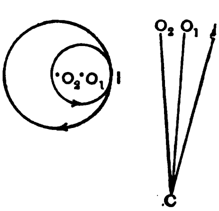
The plane through the instantaneous axis and the axis of the fixed cone passes through the axis of the rolling cone and turns about the axis of the fixed cone with the angular velocity ω_2. The motion of this plane is called the **precession** and ω_2 is the angular velocity of the precession, or, as it is sometimes called, the rate of precession.

If α_2 is zero, the fixed cone becomes a cylinder. If α_1 is zero, the rolling cone becomes a cylinder. If both are zero, we have the case of the preceding Article.



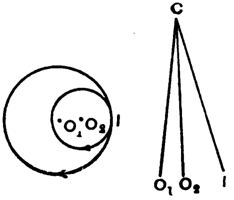
If α_2 is less than 90° and α_1 is less than 90°, we have a convex cone rolling on a convex cone, and looking from C along the axes CO_2 and CI the precession and rotation about the instantaneous axis are both clockwise or both counter-clockwise. This is called *positive* precessional rotation. It is the case of a pair of bevel-gear wheels, or of a spinning top whose point is at rest.

If α_2 is a right angle, the fixed cone becomes a flat disk with centre at C. If α_1 is a right angle, the rolling cone becomes a flat disk with centre at C. If α_2 is a right angle and α_1 is zero, we have a cylinder rolling on a plane.



If α_2 is obtuse, we have a convex cone rolling inside a concave one, and looking from C along the axes CO_2 and CI, if the precession is counter-clockwise the rotation about CI is clockwise or *vice versa*. This is *negative* precession. It is the case of the precessional motion of the earth's axis.

If α_1 is obtuse, the rolling cone becomes concave and we have a concave cone rolling on a convex cone. This is also positive precession.



The path described by a point relatively to the fixed axis is called a **spherical epitrochoid** or **hypotrochoid** according as the rolling cone is outside or inside of the fixed cone. The special form of spherical epitrochoid or hypotrochoid described by a point in the surface of the rolling cone is a **spherical epicycloid** or **hypocycloid**.

Analytical Determination of Resultant Angular Velocity and Velocity of Translation for a Rigid System with Any Number of Non-concurring Angular Velocities.* — (Compare Statics — Non-concurring Forces.) Take any point O of the rigid system as the origin of a system of rectangular co-ordinates. Let the component angular velocities be ω_1, ω_2, etc., making the angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, etc., with the axes of X, Y, Z, respectively.

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* Angular accelerations are treated in precisely the same way as angular velocities, and every equation in this Article can have ω replaced by α. The student should make such substitution and interpret the results.

Resultant Angular Velocity.—We have, just as on page 65, replacing v by ω, for the component angular velocities parallel to X, Y, Z,

$$\left.\begin{aligned} \omega_x &= \omega_1 \cos \alpha_1 + \omega_2 \cos \alpha_2 + \ldots = \Sigma \omega \cos \alpha ; \\ \omega_y &= \omega_1 \cos \beta_1 + \omega_2 \cos \beta_2 + \ldots = \Sigma \omega \cos \beta ; \\ \omega_z &= \omega_1 \cos \gamma_1 + \omega_2 \cos \gamma_2 + \ldots = \Sigma \omega \cos \gamma . \end{aligned}\right\} \quad . \; . \; . \; . \qquad (1)$$

The resultant angular velocity is

$$\omega_r = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}, \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (2)$$

and its direction cosines are

$$\cos d = \frac{\omega_x}{\omega_r}, \quad \cos e = \frac{\omega_y}{\omega_r}, \quad \cos f = \frac{\omega_z}{\omega_r}. \quad . \; . \; . \; . \qquad (3)$$

The magnitude and direction of the resultant angular velocity are thus determined.

Resultant Velocity of Translation.—Let (x_1, y_1, z_1), (x_2, y_2, z_2), etc., be the co-ordinates of points *on the rotation axes* of ω_1, ω_2, etc. We can resolve each angular velocity ω_1, ω_2, etc. (page 190), into an equal angular velocity about a parallel axis through the origin O, and a velocity of translation of the system due to a couple, given by the moment of ω_1, ω_2, etc., with reference to O. We can thus reduce the given angular velocities to a resultant angular velocity ω_r *about an axis through the origin* O, whose magnitude and direction are given by (1), (2) and (3), and a velocity of translation v_r of the axis through O. The components v_x, v_y, v_z of this velocity of the axis through O, along the axes of X, Y, Z, are therefore given by (compare Statics—Non-concurring Forces)

$$\left.\begin{aligned} M_x = v_x &= \Sigma \omega y \cos \gamma - \Sigma \omega z \cos \beta ; \\ M_y = v_y &= \Sigma \omega z \cos \alpha - \Sigma \omega x \cos \gamma ; \\ M_z = v_z &= \Sigma \omega x \cos \beta - \Sigma \omega y \cos \alpha . \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \qquad (4)$$

For any other point P whose co-ordinates are (x', y', z') we have simply to put $x - x'$, $y - y'$, $z - z'$ in place of x, y, z in (4) and we have for the components of the velocity of this point along the axes

$$\left.\begin{aligned} V_x &= \Sigma \omega y \cos \gamma - \Sigma \omega z \cos \beta + (\omega_y z' - \omega_z y') ; \\ V_y &= \Sigma \omega z \cos \alpha - \Sigma \omega x \cos \gamma + (\omega_z x' - \omega_x z') ; \\ V_z &= \Sigma \omega x \cos \beta - \Sigma \omega y \cos \alpha + (\omega_x y' - \omega_y x') . \end{aligned}\right\} \quad . \; . \; . \qquad (5)$$

Let us write

$$\left.\begin{aligned} v_x' &= \omega_y z' - \omega_z y' ; \\ v_y' &= \omega_z x' - \omega_x z' ; \\ v_z' &= \omega_x y' - \omega_y x' . \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (6)$$

Then we can write in general for the components of the linear velocity of any point whose co-ordinates are (x', y', z')

$$\left.\begin{aligned} V_x &= v_x + v_x' ; \\ V_y &= v_y + v_y' ; \\ V_z &= v_z + v_z' ; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (7)$$

where v_x, v_y, v_z are given by (4) and v_x', v_y', v_z' by (6).

If the resultant axis of rotation passes through the origin O, we have $v_x = 0$, $v_y = 0$, $v_z = 0$. Therefore equations (6) give the components of the linear velocity of the point P due to rotation about an axis through the origin O parallel to the resultant axis.

The resultant linear velocity for any point is then in general

$$V_r = \sqrt{V_x^2 + V_y^2 + V_z^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and its direction-cosines are

$$\cos a = \frac{V_x}{V_r}, \quad \cos b = \frac{V_y}{V_r}, \quad \cos c = \frac{V_z}{V_r}. \quad . \quad . \quad . \quad . \quad (9)$$

The magnitude and direction of the resultant linear velocity of any point are thus determined.

Conditions of Rest.*—If the system is at rest we must have, 1st, V_x, V_y, V_z equal to zero, or, from (7), $v_x = 0$, $v_y = 0$, $v_z = 0$; and also, 2d, $v'_x = 0$, $v'_y = 0$, $v'_z = 0$. We see from (6) that the second condition is fulfilled when $\omega_x = 0$, $\omega_y = 0$, $\omega_z = 0$, that is, when $\omega_r = 0$ or there is no rotation. In this case all the angular velocities must reduce to two *equal and opposite* resultant angular velocities. The first condition is fulfilled when equations (4) are zero. That is, the two equal and opposite resultant angular velocities must *pass through the same point*, so that their moment is zero at any point.

We have then for the equations of condition of rest, from (1),

$$\left.\begin{aligned} \Sigma\omega\cos\alpha &= 0; \\ \Sigma\omega\cos\beta &= 0; \\ \Sigma\omega\cos\gamma &= 0; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and, from (4)

$$\left.\begin{aligned} \Sigma\omega y\cos\gamma - \Sigma\omega z\cos\beta &= 0; \\ \Sigma\omega z\cos\alpha - \Sigma\omega x\cos\gamma &= 0; \\ \Sigma\omega x\cos\beta - \Sigma\omega y\cos\alpha &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If equations (11) only are fulfilled, then the two opposite resultant angular velocities pass through the origin, which is therefore at rest; but unless (10) is also fulfilled they are not equal, and we have rotation about an axis through O, but no translation.

If equations (10) only are fulfilled, there is no rotation, the two resultant angular velocities are opposite and equal, but unless (11) is also fulfilled they do not pass through the same point. Hence they form a couple, and we have translation and no rotation.

Condition that the Angular Velocities shall Reduce to a Single Angular Velocity.—If the angular velocities, then, all intersect in one point of the system, the moment at that point is zero. It has therefore no translation and the system rotates about an axis through that point. If the angular velocities do not intersect in a single point, we have in general translation and rotation.

There is, however, one case in which the angular velocities may not all intersect in one point, and yet we may have rotation only without translation. In this case the angular velocities must reduce to three, *any two of which intersect, while the other, although it does not pass through their point of intersection, yet intersects their resultant.*

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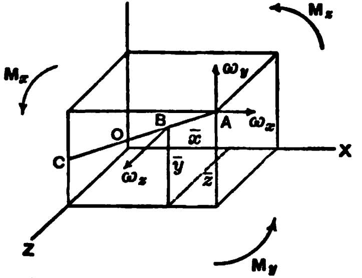
* Compare Statics—Non-concurring Forces.



Thus let the resultant angular velocities ω_x, ω_y intersect in a point A. We can then take them as acting at any point in their resultant AC.

Let ω_z intersect AC at B. Then we can take all three acting at B, and we thus have rotation only, about an axis through B.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of B. Then, since we can take ω_x, ω_y, ω_z at B, we have for the components of the velocity of the origin O

$$\left.\begin{aligned} M_x &= v_x = \omega_z\bar{y} - \omega_y\bar{z}; \\ M_y &= v_y = \omega_x\bar{z} - \omega_z\bar{x}; \\ M_z &= v_z = \omega_y\bar{x} - \omega_x\bar{y}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

If we multiply the first of these by ω_x, the second by ω_y, and the third by ω_z and add, we have (compare Statics—Non-concurring Forces)

$$v_x\,\omega_x + v_y\,\omega_y + v_z\,\omega_z = 0. \quad . \; . \; . \; . \; . \; . \; . \quad (13)$$

We should obtain the same result for any other two components intersecting and a third passing through a point on their resultant.

Equation (13) then gives the condition that all the angular velocities acting upon the system reduce to a single angular velocity at a point whose co-ordinates are $\bar{x}$, $\bar{y}$ and $\bar{z}$, and we have rotation only (page 179).

We have evidently for the equations of the projection of the line of the resultant on the co-ordinate planes

$$y = \frac{\omega_y}{\omega_x}x - \frac{v_z}{\omega_x}, \quad x = \frac{\omega_x}{\omega_z}z - \frac{v_y}{\omega_z}, \quad z = \frac{\omega_z}{\omega_y}y - \frac{v_x}{\omega_y}.$$

Parallel Velocities.—(Compare Statics—Non-concurring Forces.) If the axes of all the angular velocities are parallel, we have α, β, γ constant and the same for all. Hence from (3) and (1)

$$\left.\begin{aligned} \omega_x &= \omega_r \cos d = \cos\alpha\Sigma\omega; \\ \omega_y &= \omega_r \cos e = \cos\beta\Sigma\omega; \\ \omega_z &= \omega_r \cos f = \cos\gamma\Sigma\omega. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (14)$$

The resultant ω_r must have the common direction of the parallel components, or

$$d = \alpha, \quad e = \beta, \quad f = \gamma, \quad \text{and} \quad \omega_r = \Sigma\omega; \quad . \; . \; . \; . \quad (15)$$

that is, the resultant angular velocity is equal to the algebraic sum of all the parallel components and is parallel to them.

For any point of the system whose co-ordinates are x', y', z', we have from (4), by putting $(x - x')$, $(y - y')$, $(z - z')$ in place of x, y, z and taking α, β, γ as constant,

$$\left.\begin{aligned} V_x &= \cos\gamma\Sigma\omega(y - y') - \cos\beta\Sigma\omega(z - z') = \cos\gamma[\Sigma\omega y - y'\Sigma\omega] - \cos\beta[\Sigma\omega z - z'\Sigma\omega]; \\ V_y &= \cos\alpha\Sigma\omega(z - z') - \cos\gamma\Sigma\omega(x - x') = \cos\alpha[\Sigma\omega z - z'\Sigma\omega] - \cos\gamma[\Sigma\omega x - x'\Sigma\omega]; \\ V_z &= \cos\beta\Sigma\omega(x - x') - \cos\alpha\Sigma\omega(y - y') = \cos\beta[\Sigma\omega x - x'\Sigma\omega] - \cos\alpha[\Sigma\omega y - y'\Sigma\omega]. \end{aligned}\right\} . \quad (16)$$

If we substitute (16) and (14) in (13), we see that equation (13) is satisfied. We have therefore a single resultant velocity and rotation of the system about a fixed point. This point is given by the values of x', y', z' which make V_x, V_y, V_z zero. The point therefore whose co-ordinates are

$$\bar{x} = \frac{\Sigma\omega x}{\Sigma\omega}, \quad \bar{y} = \frac{\Sigma\omega y}{\Sigma\omega}, \quad \bar{z} = \frac{\Sigma\omega z}{\Sigma\omega}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

is at rest and the resultant axis must pass through it. This point is called the *centre of parallel velocities* (page 193). Any other point has a velocity given by (16). If $\Sigma\omega = 0$, the resultant axis is at an infinite distance, or there is no rotation, but translation only, given by (16).

Components of Motion of a Rigid System.—In order that the motion of a rigid system at any instant may be known, it is sufficient to know the velocity at that instant of some given point of the system, and the rotation of the system at that instant.

Take the given point *always as the origin*. Then the velocity of this point is known when its components V_x, V_y, V_z along the axes are given, and the rotation is known when the components ω_x, ω_y, ω_z of the angular velocity along the axes are given.

The motion of the system at any instant is then known when these six quantities, V_x, V_y, V_z, ω_x, ω_y, ω_z are given. These six quantities are called the *components of motion* of the system.

Equivalent Screw.—(Compare Statics—Non-concurring Forces.) The motion of a rigid system being thus known, it is required to find the screw motion to which it is equivalent. That is, to find the central axis, the linear velocity along the central axis, and the angular velocity about it.

Since V_x, V_y, V_z, ω_x, ω_y, ω_z are given, we have:

(1) The angular velocity about the central axis

$$\omega_r = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(2) The direction cosines of the central axis

$$\cos d = \frac{\omega_x}{\omega_r}, \quad \cos e = \frac{\omega_y}{\omega_r}, \quad \cos f = \frac{\omega_z}{\omega_r}. \quad . \quad . \quad . \quad . \quad (2)$$

(3) The linear velocity of every point resolved in a direction parallel to the central axis must be the same and equal to that along the central axis. Let u_r be the resultant linear velocity of every point of the system along the central axis, and let its components along the co-ordinates axes be u_x, u_y, u_z.

Take the point for which V_x, V_y, V_z are given, as the origin, and let the co-ordinates of any point of the central axis be x'', y'', z''. Then the components v_x, v_y, v_z of the velocity of the origin due to rotation about the central axis are, from equations (1), page 179,

$$\left.\begin{aligned} v_x &= \omega_z y'' - \omega_y z''; \\ v_y &= \omega_x z'' - \omega_z x''; \\ v_z &= \omega_y x'' - \omega_x y''. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have then

$$V_x = u_x + v_x, \quad V_y = u_y + v_y, \quad V_z = u_z + v_z,$$

or

$$u_x = V_x - v_x, \quad u_y = V_y - v_y, \quad u_z = V_z - v_z. \quad . \quad . \quad . \quad . \quad (4)$$

Hence

$$u_r = (V_x - v_x)\cos d + (V_y - v_y)\cos e + (V_z - v_z)\cos f. \qquad (5)$$

Inserting the values of the direction-cosines of the central axis from (2), we obtain

$$u_r\,\omega_r = (V_x - v_x)\omega_x + (V_y - v_y)\omega_y + (V_z - v_z)\omega_z.$$

But since $v_x\omega_x + v_y\omega_y + v_z\omega_z = 0$, this becomes

$$u_r\,\omega_r = V_x\omega_x + V_y\omega_y + V_z\omega_z. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We also have from (4)

$$u_r\cos d = u_x = V_x - v_x,\quad u_r\cos e = V_y - v_y,\quad u_r\cos f = V_z - v_z. \qquad (7)$$

Hence from (2) and (3)

$$\frac{u_r}{\omega_r} = \frac{V_x + \omega_y z'' - \omega_z y''}{\omega_x} = \frac{V_y + \omega_z x'' - \omega_x z''}{\omega_y} = \frac{V_z + \omega_x y'' - \omega_y x''}{\omega_z}. \qquad (8)$$

Equations (8) give the equation of the central axis.

From (6) and (1) we have

$$\frac{u_r}{\omega_r} = \frac{V_x\omega_x + V_y\omega_y + V_z\omega_z}{\omega_x^2 + \omega_y^2 + \omega_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This we have called the *unit pitch* (page 191), or the distance of advance during a rotation of one radian about the central axis.

If we substitute (9) in (8) and reduce, we have for the equation of the central axis

$$\begin{aligned}\frac{1}{\omega_x}\left(x'' - \frac{V_z\omega_y - V_y\omega_z}{\omega_r^2}\right) &= \frac{1}{\omega_y}\left(y'' - \frac{V_x\omega_z - V_z\omega_x}{\omega_r^2}\right)\\ &= \frac{1}{\omega_z}\left(z'' - \frac{V_y\omega_x - V_x\omega_y}{\omega_r^2}\right). \quad . \quad (10)\end{aligned}$$

Therefore the central axis passes through a point whose co-ordinates are *

$$x'' = \frac{V_z\omega_y - V_y\omega_z}{\omega_r^2},\quad y'' = \frac{V_x\omega_z - V_z\omega_x}{\omega_r^2},\quad z'' = \frac{V_y\omega_x - V_x\omega_y}{\omega_r^2}. \qquad (11)$$

If we substitute these values of x'', y'', z'', in (3) and (7), we have from (2)

$$\left.\begin{aligned}V_x &= u_r\cos d - \omega_r(z''\cos e - y''\cos f), & \omega_x &= \omega_r\cos d;\\ V_y &= u_r\cos e - \omega_r(x''\cos f - z''\cos d), & \omega_y &= \omega_r\cos e;\\ V_z &= u_r\cos f - \omega_r(y''\cos d - x''\cos e), & \omega_z &= \omega_r\cos f.\end{aligned}\right\}. \qquad (12)$$

When, therefore, the components of motion V_x, V_y, V_z, ω_x, ω_y, ω_z are given for any point, we find ω_r from (1), the direction of the central axis from (2), and the position of the central axis with reference to that point as an origin from (11). We have also the velocity of advance u_r from (9).

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* Since velocity in the hodograph is normal acceleration in the path (page 52), $V_z\omega_y - V_y\omega_z$ is the component in the direction of X of the normal linear acceleration of the origin due to rotation about the central axis. The normal linear acceleration is $p\omega_r^2$. Hence $\frac{V_z\omega_y - V_y\omega_z}{\omega_r^2}$ is the projection of p on the axis of X.

On the other hand, if the position of the central axis (x'', y'', z'') is known together with the linear velocity u_r along it and the angular velocity ω_r round it, the components of the motion for the origin are given by (12).

The Invariant.—(Compare Statics—Non-concurring Forces.) From (6) we see that the quantity

$$V_x\omega_x + V_y\omega_y + V_z\omega_z$$

is always equal to $u_r\omega_r$, and is therefore invariable no matter what point is taken and whatever the values of ω_x, ω_y, ω_z, that is, whatever the direction of the axes. This quantity is therefore called the **invariant** of the components. Since ω_r is also invariable whatever point is taken and whatever the direction of the axes, it may be called the invariant of the rotation.

If the motion is such that the invariant is zero, it follows that either $u_r = 0$ or $\omega_r = 0$. The condition

$$V_x\omega_x + V_y\omega_y + V_z\omega_z = 0$$

is therefore the condition that the motion *is equivalent to either a simple translation or a simple rotation.* If ω_r is not zero and this condition is fulfilled, we have rotation only (see pages 179, 200).

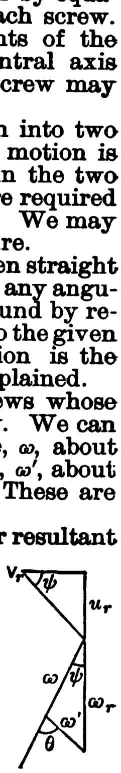
Composition and Resolution of Screws.—(Compare Statics—Non-concurring Forces.) If two screws are given, then by equation (12) we can find the six components of motion of each screw. Adding these two and two, we have the six components of the resultant screw. Then by (1), (2), (6) and (11) the central axis together with the linear and angular velocities of the screw may be found.

Conversely, we may resolve any given screw motion into two screws in an infinite number of ways. Since a screw motion is represented by six components at any point, we have in the two screws twelve quantities at our disposal. Six of these are required to make the two screws equivalent to the given screw. We may therefore in general satisfy six other conditions at pleasure.

Thus we may choose the axis of one screw to be any given straight line we please with any given linear velocity along it and any angular velocity round it. The other screw may then be found by reversing this assumed screw and joining it thus changed to the given motion. The screw equivalent to this compound motion is the second screw, and it may be found in the manner just explained.

Or again, we may represent the motion by two screws whose unit pitches are both zero, the axis of one being arbitrary. We can thus represent any motion by two angular velocities, one, ω, about an axis which we may choose at pleasure, and the other, ω', about some axis which does not in general cut the first axis. These are called **conjugate axes.**

These angular velocities are such that ω_r would be their resultant if their axes were placed parallel to their actual positions, so as to intersect the central axis. If then d is the shortest distance between the axes, we have $V_r = d\omega$; and if ψ is the angle between ω and ω_r, and θ the angle between ω and ω', we have



$$V_r \sin\psi = u_r, \qquad \text{or} \quad \sin\psi = \frac{u_r}{V_r}.$$

Also

$$\omega_r \sin\psi = \omega' \sin\theta, \quad \text{or} \quad \sin\psi = \frac{\omega' \sin\theta}{\omega_r}.$$

Hence

$$V_r\omega'\sin\theta = u_r\omega_r, \quad \text{or} \quad d\omega\omega'\sin\theta = u_r\omega_r;$$

hence

$$d = \frac{u_r\omega_r}{\omega\omega'\sin\theta}. \qquad \text{(1)}$$

EXAMPLES.

(1) *A line DE moves, keeping its extremities in two fixed lines ADB, AEC. Find the instantaneous centre and the direction of motion of any point G at any instant.*

Ans. From D and E draw DF and EF perpendicular to AB and AC, meeting at F. Then F is the instantaneous centre (page 189). Join GF. The direction of motion of G is perpendicular to GF.

(2) *A line DE moves, keeping its extremities in two fixed lines, one, ADB, vertical and the other, AEC, horizontal, and makes at a given instant an angle of 30° with the horizontal. Find (a) the direction of motion of the middle point of DE at the instant, and (b) the point whose motion is inclined at that instant 30° to AC.*

Ans. (a) Inclined 60° to AC; (b) $\frac{1}{4}DE$ from E.

(3) *A line moves so that its extremities remain in a given circle. Find the instantaneous centre of motion for any instant.*

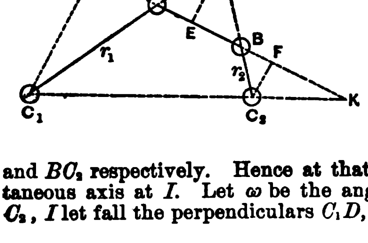
Ans. The centre of the circle.

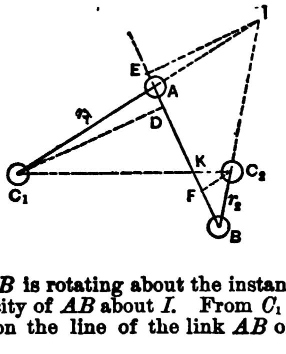
(4) *Find the ratio of the velocity of any point of a screw to its velocity of advance.*

Ans. $\sqrt{\frac{d^2+4\pi^2r^2}{d}}$, where d is the pitch, r the radius of the screw (page 191).

(5) *Let C_1 and C_2 be fixed axes about which turn the cranks CA_1, C_2B, whose free ends are connected by the link AB, jointed at A and B. The axes are perpendicular and the plane of motion parallel to the paper. If the linear and angular velocities of A are v_1, ω_1, find the linear and angular velocities v_2, ω_2 of B.*

Ans. Let $AC_1 = r_1$ and $BC_2 = r_2$. Produce C_1A and C_2B to meet in I. Then at any instant the linear velocities of A and B are perpendicular to AC_1





and BC_2 respectively. Hence at that instant AB is rotating about the instantaneous axis at I. Let ω be the angular velocity of AB about I. From C_1, C_2, I let fall the perpendiculars C_1D, C_2F, IE on the line of the link AB or its prolongation. Also draw the line of centres C_1C_2 cutting the link, prolonged, if necessary, in the point K. Then

$$r_1\omega_1 = AI.\,\omega, \quad \text{or} \quad \frac{\omega_1}{\omega} = \frac{AI}{r_1} = \frac{IE}{C_1D};$$

$$r_2\omega_2 = BI.\,\omega, \quad \text{or} \quad \frac{\omega_2}{\omega} = \frac{BI}{r_2} = \frac{IE}{C_2F}.$$

Since $r_1\omega_1 = v_1$ and $r_2\omega_2 = v_2$, we have

$$\frac{v_2}{v_1} = \frac{BI}{AI}; \quad \text{also,} \quad \frac{\omega_2}{\omega_1} = \frac{C_1D}{C_2F} = \frac{C_1K}{C_2K}.$$

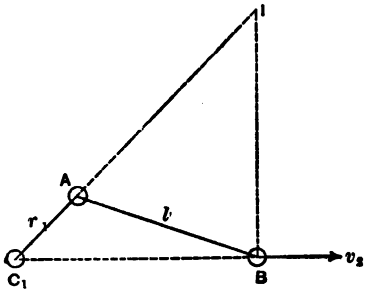
Hence—

1. The linear velocities of B and A are to each other as their distances from the instantaneous axis.

2. The angular velocities of the cranks are to each other inversely as the perpendiculars from their centres of motion upon the line of the link; or inversely as the segments into which the line of the link cuts the line of centres.

(6) In the case of the crank and connecting rod, since B moves in a straight line C_1B, we have BI always perpendicular to C_1B, and hence

$$\frac{v_2}{v_1} = \frac{BI}{AI}, \quad \text{or} \quad v_2 = \frac{BI}{AI} r_1\omega_1.$$



Let the distance $C_1B = s$, the length of the connecting-rod $= l$, and the angle of the crank r_1 with $C_1B = \theta_1$. Then we have

$$BI = s\tan\theta_1, \quad AI = \frac{s}{\cos\theta_1} - r_1,$$

$$s = r_1\cos\theta_1 + \sqrt{l^2 - r_1^2\sin^2\theta_1};$$

or, if l is very long compared to r_1, approximately

$$s = r_1\cos\theta_1 + l - \frac{r_1^2\sin^2\theta_1}{2l}.$$

Hence

$$v_2 = \frac{s\tan\theta_1 \,.\, r_1\omega_1}{\dfrac{s}{\cos\theta_1} - r_1} = \frac{s\sin\theta_1}{s - r_1\cos\theta_1} r_1\omega_1.$$

When $\theta_1 = 90$, we have $v_2 = r_1\omega_1 = v_1$, or the velocity of A and B are equal, and BI and AI are infinite. When $\theta_1 = 0$ or 180°, we have $v_2 = 0$ and $s = l + r_1$ or $l - r_1$. These are the "*dead points*" of the crank, or the ends of the stroke.

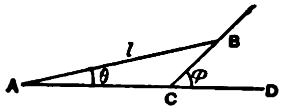
(7) *A rod (length $= l$) hangs by a small ring at its upper end from a fixed horizontal rod. To the former an angular velocity ω is given in a vertical plane through the fixed rod, so that the centre of the movable rod moves vertically. Find the linear velocity of its centre when its inclination to the vertical is θ.*

Ans. $\omega l \sin\theta$.

(8) *A disk (radius $= r$) rolls without sliding on a plane. Find the relation between its angular velocity ω and the linear velocity v of its centre.*

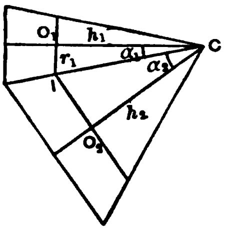
Ans. The point of contact with the plane is at rest at any instant, or $r\omega = -v$.

(9) *A rod AB (length = l) rotates about a hinge at A and rests with its end B on the surface of a wedge BCD. The wedge advances towards A with velocity v. The angles BAC = θ, BCD = φ. Find the angular velocity ω of the rod.*



Ans. $\omega = \frac{v}{l} \cdot \frac{\sin \phi}{\cos (\phi - \theta)}$.

(10) *Two bevel-gear wheels have the angle between their axes 70°. The rolling wheel is required to make 3½ revolutions about its axis while going around the axis of the fixed wheel once. Find the angles of the bevel. If the inner radius of the fixed wheel is 50 inches, find that of the rolling wheel and the length of the axes.*



Ans. (See page 196.) The angular-velocity ratio $\frac{\omega_1}{\omega_2} = \frac{7}{2}$. Hence

$$\text{tang}\ \alpha_2 = \frac{7 \sin 70^\circ}{2 + 7 \cos 70^\circ}, \quad \text{or} \quad \alpha_2 = 56^\circ\ 15',$$

and hence $\alpha_1 = 13^\circ\ 45'$. We have also $r_1 = \frac{100}{7}$ inches, and $h_2 = \dfrac{\frac{100}{7} + 50 \cos 70^\circ}{\sin 70} = 33.4$ inches,

$$h_1 = \frac{50 + \frac{100}{7} \cos 70^\circ}{\sin 70^\circ} = 58.4 \text{ inches.}$$

(11) *The angle between the plane of the earth's equator and the plane of the ecliptic is 23° 27′ 28″. The earth rotates about its polar axis in one sidereal day and makes a revolution about the axis perpendicular to the plane of the ecliptic in 25868 years. Find the instantaneous axis.*

Ans. (Page 196.) We have

$$\omega_1 = 2\pi \text{ radians per day,} \quad \text{and} \quad \omega_2 = \frac{2\pi}{25868 \times 365\frac{1}{4}} \text{ radians per day.}$$

Also the angle $O_2CO_1 = \alpha_1 + \alpha_2 = 23^\circ\ 27'\ 28''$. Therefore O_1I is

$$O_1I = r \tan \alpha_1 = \frac{r \sin (\alpha_1 + \alpha_2)}{\frac{\omega_1}{\omega_2} + \cos (\alpha_1 + \alpha_2)},$$

where r is the polar radius of the earth, or 3950 miles. The radius of the rolling cone is then $O_1I = 5.52$ ft. and the angle $\alpha_1 = 0''.00867$.

(13) *A rigid system has an angular velocity of $\omega_1 = 40$ radians per sec. about an axis parallel to the axis of X, passing through a point whose co-ordinates are $x_1 = 2$ ft., $y_1 = 3$ ft., $z_1 = 0$, and a simultaneous angular velocity of $\omega_2 = 30$ radians per sec. about an axis coinciding with the axis of Y. Find the resultant angular velocity and the instantaneous axis.*

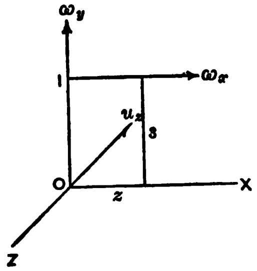
Ans. (Page 198.) We have $\omega_x = +40$, $\omega_y = +30$, $\omega_z = 0$, $\omega_r = 50$ radians per sec. The instantaneous axis makes the angles with the axes given by

$$\cos d = \frac{40}{50}, \quad \cos e = \frac{30}{50}, \quad \cos f = 0;$$

$$\text{or} \quad d = 36^\circ\ 52', \quad e = 53^\circ\ 7', \quad f = 0^\circ.$$

The velocity of translation of the system is given by

$u_x = 0, \quad u_y = 0, \quad u_z = -120$ ft. per sec.



The condition $u_x\omega_x + u_y\omega_y + u_z\omega_z = 0$ is satisfied. Therefore there is rotation only about the instantaneous axis which passes through the intersection I of ω_x and ω_y.

The velocity of any point whose co-ordinates are $x' = 2$, $y' = 2$, $z' = 3$ is given by

$V_x = \omega_y z' = +90$ ft. per sec.;

$V_y = -\omega_x z' = -120$ ft. per sec.;

$V_z = u_z - \omega_y x' + \omega_x y' = -100$ ft. per sec.

The resultant velocity of this point is then $V_r = 180$ ft. per sec. and its direction cosines are

$$\cos a = \frac{90}{180}, \quad \cos b = \frac{120}{180}, \quad \cos c = \frac{100}{180};$$

$$\text{or} \quad a = 60^\circ, \quad b = 131^\circ\ 48', \quad c = 123^\circ\ 45'.$$

(13) ***A rigid system has the angular velocities***

$\omega_1 = 50, \quad \omega_2 = 30, \quad \omega_3 = 70, \quad \omega_4 = 90,$ and $\omega_5 = 120$ radians per sec.

about axes passing through points of the rigid system given by

$$x_1 = +\ 5 \text{ ft.}, y_1 = +10 \text{ ft.}; \quad x_2 = +\ 9 \text{ ft.}, y_2 = +12 \text{ ft.};$$
$$x_3 = +17 \text{ ft.}, y_3 = +14 \text{ ft.}; \quad x_4 = +20 \text{ ft.}, y_4 = +13 \text{ ft.};$$
$$x_5 = +15 \text{ ft.}, y_5 = +\ 8 \text{ ft.};$$

and making with the co-ordinate axes the angles

$$\alpha_1 = 70^\circ, \beta_1 = 20^\circ; \quad \alpha_2 = 60^\circ, \beta_2 = 150^\circ; \quad \alpha_3 = 120^\circ, \beta_3 = 0^\circ;$$
$$\alpha_4 = 150^\circ, \beta_4 = 120^\circ; \quad \alpha_5 = 90^\circ, \beta_5 = 0^\circ.$$

Find the resultant, etc. (Compare Vol. II, Statics.)

Ans. (Page 198.) We have for the components of the angular velocities parallel to the axes

$\omega_x = 50 \cos 70^\circ + 30 \cos 60^\circ - 70 \cos 60^\circ - 90 \cos 30^\circ = -80.842$ rad. per sec.;

$\omega_y = 50 \cos 20^\circ - 30 \cos 30^\circ + 120 + 70 \cos 30^\circ - 90 \cos 60^\circ = +156.626$ rad. [per sec.;

$\omega_z = 0.$

The resultant angular velocity is given in magnitude by

$$\omega_r = \sqrt{\omega_x^2 + \omega_y^2} = +176.259 \text{ radians per sec.},$$

and its direction-cosines by

$$\cos d = \frac{\omega_x}{\omega_r} = \frac{-80.842}{176.259}, \quad \text{or} \quad d = 117^\circ\ 18'\ 1'';$$

$$\cos e = \frac{\omega_y}{\omega_r} = \frac{+156.626}{176.259}, \quad \text{or} \quad e = 27^\circ\ 18'\ 1''.$$

We have from equation (4), page 198,

$$\Sigma\omega x \cos\beta = +50 \cos 20^\circ \times 5 - 30 \cos 30^\circ \times 9 + 70 \cos 30^\circ \times 17$$
$$- 90 \cos 60^\circ \times 20 + 120 \times 15 = +1931.67 \text{ ft. per sec.};$$

$$\Sigma\omega y \cos\alpha = +50 \cos 70^\circ \times 10 + 30 \cos 60^\circ \times 12 - 70 \cos 60^\circ \times 14$$
$$- 90 \cos 30^\circ \times 13 = -1152.245 \text{ ft. per sec.}$$

We have then for the components of the linear velocity of the origin

$$v_x = 0, \quad v_y = 0, \quad v_z = \Sigma \omega x \cos\beta - \Sigma \omega y \cos\alpha = +3083.915 \text{ ft. per sec.}$$

Since then equation (13), page 200,

$$v_x\omega_x + v_y\omega_y + v_z\omega_z = 0$$

is satisfied, the angular velocities reduce to a single resultant angular velocity and we have rotation only.

The moment of this resultant angular velocity relative to the origin gives us

$$v_r = \sqrt{v_x^2 + v_y^2 + v_z^2} = v_z = +3083.915 \text{ ft. per sec.}$$

Its lever-arm is

$$r = \frac{v_r}{\omega_r} = \frac{3083.915}{176.259} = 17.5 \text{ ft.}$$

The equation of the line of direction of the resultant angular velocity is

$$y = \frac{\omega_y}{\omega_x}x - \frac{v_z}{\omega_x} = -1.95x + 38.14.$$

The co-ordinates of the point through which this resultant angular velocity passes are given from equations (17), page 201:

$$\bar{x} = \frac{\Sigma \omega x \cos\beta}{\omega_y} = +12\tfrac{1}{4} \text{ ft.}; \quad \bar{y} = \frac{\Sigma \omega y \cos\alpha}{\omega_x} = +14.25 \text{ ft.}$$

(14) *Find the resultant angular velocity for a number of parallel angular velocities given by*

$$\begin{array}{lll}
\omega_1 = +\ 33 \text{ rad. per sec.}; & x_1 = +25 \text{ ft.}; & y_1 = +13 \text{ ft.}; \\
\omega_2 = +\ 20 \text{ `` `` ``} & x_2 = -10 \text{ ``} & y_2 = -15 \text{ ``} \\
\omega_3 = -\ 35 \text{ `` `` ``} & x_3 = +15 \text{ ``} & y_3 = -27 \text{ ``} \\
\omega_4 = -\ 72 \text{ `` `` ``} & x_4 = -31 \text{ ``} & y_4 = +17 \text{ ``} \\
\omega_5 = +120 \text{ `` `` ``} & x_5 = +23 \text{ ``} & y_5 = -19 \text{ ``}
\end{array}$$

Ans. $\omega_r = +66$ radians per sec.; $\bar{x} = +77.15$ ft.; $\bar{y} = -36.82$ ft.

(15) *Find the resultant, etc., for the angular velocities given by*

$$\omega_1 = 50,\ \omega_2 = 70,\ \omega_3 = 90,\ \omega_4 = 120 \text{ radians per sec.}$$

$$\begin{array}{llllll}
\alpha_1 = 60^\circ; & \beta_1 = 40^\circ; & \gamma_1 \text{ acute}; & x_1 = 0; & y_1 = 0; & z_1 = 0; \\
\alpha_2 = 65^\circ; & \beta_2 = 45^\circ; & \gamma_2 \text{ acute}; & x_2 = +1 \text{ ft.}; & y_2 = +4 \text{ ft.}; & z_2 = +7 \text{ ft.}; \\
\alpha_3 = 70^\circ; & \beta_3 = 50^\circ: & \gamma_3 \text{ acute}; & x_3 = +2 \text{ ``} & y_3 = +5 \text{ ``} & z_3 = +8 \text{ ``} \\
\alpha_4 = 75^\circ; & \beta_4 = 55^\circ; & \gamma_4 \text{ obtuse}; & x_4 = +3 \text{ ``} & y_4 = +6 \text{ ``} & z_4 = +9 \text{ ``}
\end{array}$$

(Compare Vol. II, Statics.)

Ans. We find the angles γ by the formula, page 12,

$$\cos^2\gamma = -\cos(\alpha + \beta)\cos(\alpha - \beta).$$

Then from page 198 we have

$$\omega_x = +116.423, \quad \omega_y = +214.480, \quad \omega_z = -51.057 \text{ rad. per sec.}$$

Therefore the resultant angular velocity is

$$\omega_r = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = +249.325 \text{ rad. per sec.},$$

and its direction-cosines are given by

$$\cos d = \frac{\omega_x}{\omega_r}, \quad \cos e = \frac{\omega_y}{\omega_r}, \quad \cos f = \frac{\omega_z}{\omega_r};$$

or

$$d = 62^\circ\ 9'\ 48'', \quad e = 30^\circ\ 39'\ 20'', \quad f = 101^\circ\ 49'.$$

We also have for the components of the velocity of the origin, from equations (4), page 198,

$$v_x = -1838.604, \quad v_y = +928.947, \quad v_z = -86.903 \text{ ft. per sec.}$$

The resultant linear velocity of the origin is then

$$v_r = \sqrt{v_x^2 + v_y^2 + v_z^2} = +2061.789 \text{ ft. per sec.},$$

and its direction-cosines are given by

$$\cos a = \frac{v_x}{v_r}, \quad \cos b = \frac{v_y}{v_r}, \quad \cos c = \frac{v_z}{v_r};$$

or

$$a = 153° \, 5' \, 40'', \quad b = 63° \, 14' \, 15'', \quad c = 92° \, 24' \, 56''.$$

The equations of the projections of the resultant angular velocity on the co-ordinate planes are

$$y = 1.885x + 0.746, \quad x = -2.28z + 18.19, \quad z = -0.238y - 8.57.$$

We see that $v_x\omega_x + v_y\omega_y + v_z\omega_z$ does not, in this case, equal zero. Hence (page 179) the angular velocities do not reduce to a single resultant, but to a resultant angular velocity about the central axis and translation along that axis due to an angular-velocity couple.

The resultant angular velocity about the central axis is, as already found, $\omega_r = +249.325$ rad. per sec., and its angles d, e, f with the axes are already found.

The co-ordinates of the central axis are given by equations (11), page 202:

$$x'' = \frac{\omega_y v_z - \omega_z v_y}{\omega_r^2} = +0.463 \text{ ft.}; \quad y'' = \frac{\omega_z v_x - \omega_x v_z}{\omega_r^2} = +1.673 \text{ ft.};$$

$$z'' = \frac{\omega_x v_y - \omega_y v_x}{\omega_r^2} = +8.08 \text{ ft.}$$

The resultant linear velocity u_r along the central axis is given by equation (6), page 202:

$$u_r = \frac{v_x\omega_x + v_y\omega_y + v_z\omega_z}{\omega_r} = -41.624 \text{ ft. per sec.}$$

Its direction-cosines are the same as for ω_r. The components of u_r are given by equations (7), page 202:

$$u_x = u_r \cos d = -19.481, \quad u_y = u_r \cos e = -35.806,$$
$$u_z = u_r \cos f = +8.5238 \text{ ft. per sec.}$$

(16) *In the preceding example find what the co-ordinates x_4, y_4, z_4 of the angular velocity ω_4 must be in order that all the angular velocities may reduce to a single resultant angular velocity.* (Compare Vol. II, Statics.)

Ans. We must evidently have ω_x, ω_y, ω_z, ω_r and the angles d, e, f unchanged, since changing the co-ordinates x_4, y_4, z_4 without changing the magnitude or direction of ω_4 has no effect on the magnitude or direction of the resultant ω_r.

We have then

$$\left.\begin{aligned} v_x &= -659.571 - 93.262y_4 - 68.829z_4; \\ v_y &= +369.629 + 31.059z_4 + 93.262x_4; \\ v_z &= -107.036 + 68.829x_4 - 31.059y_4. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have as the equation of condition for a single resultant

$$v_x\omega_x + v_y\omega_y + v_z\omega_z = 0,$$

or

$$116.423v_x + 214.48v_y - 51.057v_z = 0,$$

or

$$v_x + 1.842v_y - 0.4386v_z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) we obtain

$$(v_x + 659.571)\,31.059 + (v_y - 369.629)\,68.829 = (v_z + 107.086)\,93.262,$$

or

$$v_x + 2.216v_y - 3.003v_z = +481.084. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3) we obtain

$$0.374v_y - 2.564v_z = +481.084.$$

If we retain for v_y its value in the preceding example, $+928.947$ ft. per sec., we shall have

$$v_z = -52.108, \quad v_x = -1733.975 \text{ ft. per sec.}$$

If we substitute these values in (1), we obtain

$$93.262y_4 + 68.829z_4 = +1074.4;$$

$$31.059z_4 + 93.262x_4 = +\ 559.308;$$

$$68.829x_4 - 31.059y_4 = +\ \ 54.934.$$

Hence

$$x_4 = -0.333z_4 + 5.997, \quad y_4 = -0.738z_4 + 11.520.$$

If then we assume $z_4 = 0$, we have $x_4 = +5.997$, $y_4 = +11.52$ ft.

(17) *Using the values of the preceding example, find the point through which the resultant angular velocity passes.* (Compare Vol. II, Statics.)

Ans. We have $\omega_x = +116.423$, $\omega_y = +214.480$, $\omega_z = -51.057$, $\omega_r = +249.825$ radians per sec.; $d = 62°\ 9'\ 48''$, $e = 30°\ 39'\ 20''$, $f = 101°\ 49'$; $v_x = -1733.975$, $v_y = +928.947$, $v_z = -52.108$, $v_r = +1967.823$ ft. per sec.;

$$a = 151°\ 47', \quad b = 61°\ 49'\ 53'', \quad c = 91°\ 31'\ 3''.$$

The co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ are given (page 200) by

$$-1733.975 = \omega_z\bar{y} - \omega_y\bar{z} = -\ 51.057\bar{y} - 214.480\bar{z};$$

$$+\ 928.947 = \omega_x\bar{z} - \omega_z\bar{x} = +116.423\bar{z} +\ 51.057\bar{x};$$

$$-\ \ 52.108 = \omega_y\bar{x} - \omega_x\bar{y} = +214.480\bar{x} - 116.423\bar{y}.$$

Hence we obtain

$$\bar{x} = -2.2802\bar{z} + 18.194;$$

$$\bar{y} = -4.2008\bar{z} + 33.961.$$

If we assume $\bar{z} = 0$, we have $\bar{x} = +18.194$, $\bar{y} = +33.961$ ft.

If we should introduce then a fifth angular velocity, $\omega_5 = +249.825$, whose direction makes with the axes the angles

$$\alpha_5 = 117°\ 50'\ 12'', \quad \beta_5 = 149°\ 20'\ 40'', \quad \gamma_5 = 78°\ 11',$$

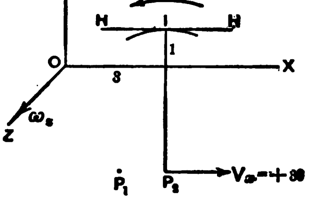
passing through a point whose co-ordinates are $x_5 = +18.194$, $y_5 = +33.961$ and $z_5 = 0$, the conditions for rest (page 199) would be satisfied, and we should have $\omega_r = 0$, $v_r = 0$.

(18) *A point of a rigid system rotates about an axis at a distance of 5 feet. The linear displacement of the point is 8 ft. Find the angular displacement and the direction of the linear displacement.*

Ans. $\sin\frac{\theta}{2} = \frac{d}{2r} = \frac{4}{5}$. θ = angular displacement $= 106°\ 14' = 1.858$ radians. The linear displacement makes an angle of 36° 53′ with the radius of rotation.

(19) *A point of a rigid system has at any instant the component linear velocities $V_x = +6$, $V_y = -18$, $V_z = +40$ ft. per sec., and the system at the same instant rotates about an axis perpendicular to the plane of XY with an angular velocity of 6 radians per sec. in the direction from X towards Y. Find the equivalent screw motion of the system.*

Ans. (Page 201.) We take the given point as the origin. Since the axis is perpendicular to the plane of XY, we have $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega_r = +6$ radians per sec.



Since the condition $V_x\omega_x + V_y\omega_y + V_z\omega_z = 0$ is not fulfilled, we have rotation and translation combined, or screw motion.

From equation (2) we have $\cos d = 0$, $\cos e = 0$, $\cos f = 1$, or the central axis is parallel to the axis of Z.

The position of the central axis is from equation (11) given by

$$x'' = +3 \text{ ft.}, \quad y'' = +1 \text{ ft.}, \quad z'' = 0.$$

It is therefore at I as shown in the figure with respect to the given point O.

Substituting these values in equation (3) we have for the components along the co-ordinate axes of the velocity of O due to rotation about the central axis

$$v_x = +6, \quad v_y = -18 \text{ ft. per sec.}, \quad v_z = 0.$$

Therefore from (4) we have for the components along the axes of the translation of O,

$$u_x = 0, \quad u_y = 0, \quad u_z = +40 \text{ ft. per sec.}$$

The system, therefore, at the instant in question rotates about an axis through I perpendicular to the plane of XY, in a direction from X towards Y, with the angular velocity of 6 radians per sec., and at the same time moves along this axis in the direction OZ with a velocity of translation of 40 ft. per sec.

If the axis of rotation and angular velocity do not change in direction or magnitude, the system advances along the central axis during a rotation of one radian, a distance equal to the *unit pitch*, given by equation (9), viz., $6\frac{2}{3}$ ft. In one complete rotation then it advances a distance of $2\pi \times 6\frac{2}{3} = 20.9$ ft. This is the *pitch* of the screw motion (page 191).

The velocity at any point, as P_1 or P_2, due to rotation about the central axis is equal to $IP_1 . \omega_z$ or $IP_2 . \omega_z$, where IP_1 or IP_2 is the radius vector or perpendicular from the point upon the central axis. If then we take P_1 as origin and the co-ordinates of I are $x = +2$ ft., $y = +5$ ft., $z = +3$ ft., we have from (3), for the components of the velocity of P_1 due to rotation about the central axis,

$$v_x = +30 \text{ ft. per sec.}, \quad v_y = -12 \text{ ft. per sec.}, \quad v_z = 0;$$

and since $u_x = 0$, $u_y = 0$, $u_z = +40$ ft. per sec., the components of the total velocity of P_1 are, from (4),

$$V_x = +30 \text{ ft. per sec.}, \quad V_y = -12 \text{ ft. per sec.}, \quad V_z = +40 \text{ ft. per sec.}$$

In the same way for the point P_2, if we take it as origin and the co-ordinates of I are $x = 0$, $y = +5$ ft., $z = 0$, we have

$$v_x = +30 \text{ ft. per sec.}, \quad v_y = 0, \quad v_z = 0;$$

$$V_x = +30 \text{ ft. per sec.}, \quad V_y = 0, \quad V_z = +40 \text{ ft. per sec.}$$

If the velocities V_x and V_z of the point P_2 do not change in direction or magnitude, we have the case of a system translated in the direction OZ and rotating about an axis through P_2, while at the same time this axis has a velocity of translation in a straight line (page 193).

The motion of the system would then be the same as if a cylindrical surface of radius $P_2I = 5$ ft., fixed to the system with its axis passing through P_2 at right angles to the plane of XY, rolled on the plane HIH parallel to the plane XY with the angular velocity $\omega_z = +6$ radians per sec., while at the same time the cylinder is translated parallel to OZ with the velocity $V_z = +40$ ft. per sec.

(20) *A base-ball rotates about an axis through its centre in a horizontal plane with an angular velocity $\omega_z = -60$ radians per sec., and its centre has a horizontal velocity of translation of $V = 50$ ft. per sec. in a direction making an angle of 36° 52′ with the axis of rotation. Find the motion of the ball.*

Ans. Let the plane of ZX be horizontal and take the centre as origin. Then, since V is in this plane, we have

$$V_x = +30 \text{ ft. per sec.}, \quad V_y = 0,$$
$$V_z = +40 \text{ ft. per sec.}$$

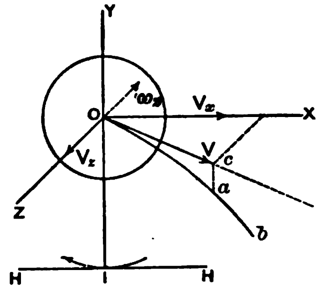
Also,

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = -60 \text{ rad. per sec.}$$

The rotation is then clockwise, or from Y towards X, as shown in the figure.

Then, just as in the preceding example, the central axis is parallel to the axis of Z, and the position of the central axis is, from equation (11), page 202, given by

$$x'' = 0, \quad y'' = OI = -\frac{1}{2} \text{ ft.}, \quad z'' = 0.$$



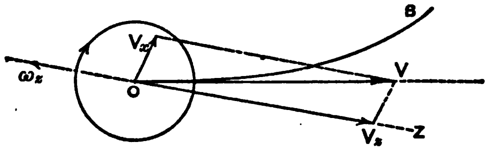
If then we neglect the acceleration due to the attraction of the earth, the motion of the ball is a screw motion consisting of a velocity $u_z = +40$ ft. per sec. along the axis of rotation OZ through the centre O of the ball, and a rotation of $\omega_z = -60$ radians per sec. about this axis, together with a translation of this axis of $V_x = +40$ ft. per sec.

Or, neglecting the acceleration due to gravity, the motion is the same (page 198) as if the ball were part of a cylinder of radius $OI = \frac{1}{2}$ ft. whose axis OZ is the axis of rotation of the ball, and this cylinder rolls on the horizontal plane HIH with angular velocity $\omega_z = -60$ radians per sec., while at the same time the cylinder is translated along OZ with the velocity $+40$ ft. per sec.

The centre O of the ball moves then in the resultant of V_x and V_z, or along the straight line OV in the horizontal plane XZ, with a velocity $V = 50$ ft. per sec., at an angle of 36° 52′ with the axis of rotation OZ.

If now, owing to gravity, the ball falls vertically while the centre moves along OV, then we must consider the plane HIH as falling vertically with the ball. The centre moves then in a curve Oab, the projection of which upon the plane XZ is a straight line Oc.

(21) *Ball-players assert that the projection of this curve Oab (preceding example) upon the plane XZ is not a straight line but a curve. Explain how this can be.*



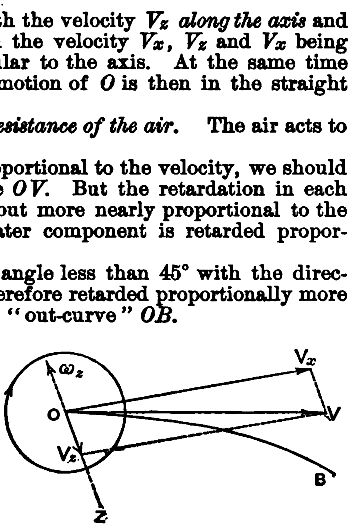
Ans. We have seen in the preceding example that if the centre O of the ball has a velocity V and at the same time the ball has an angular velocity ω_z

around an axis OZ, the centre O moves with the velocity V_z *along the axis* and at the same time the axis itself moves with the velocity V_x, V_z and V_x being the components of V along and perpendicular to the axis. At the same time the ball rotates about the axis OZ. The motion of O is then in the straight line OV.

But no account has been taken of the *resistance of the air*. The air acts to cause a retardation of V_z and V_x.

If the retardation in each case were proportional to the velocity, we should still have motion of O in the straight line OV. But the retardation in each case is not proportional to the velocities but more nearly proportional to the squares of the velocities. Hence the greater component is retarded proportionally more than the less.

If then the rotation axis OZ makes an angle less than 45° with the direction of V, V_z is greater than V_x and is therefore retarded proportionally more than V_x. The centre O moves then in an "out-curve" OB.



If, however, the axis of rotation OZ makes an angle greater than 45° with the direction of V, V_x is greater than V_z and is therefore retarded proportionally more than V_z. The centre O moves then in an "in-curve."

In either case the velocity is retarded least in the direction of least resistance and the centre swerves in the direction of the smallest component of V.

Thus by "twisting" the ball the pitcher is able to make it curve slightly by either to right or to left according as the axis of rotation makes an angle with the velocity of projection greater or less than 45°.

If the axis of rotation makes an angle of 45° with the velocity of projection, there should be no curve. If it is at right angles to the velocity of projection, there should be no curve.

The cause of curvature is thus due to the resistance of the air, but it is not, as is generally supposed, due to the ball rolling upon a cushion of compressed air in front of it, since in that case we should always have curvature in one direction for one direction of rotation.

In the first of our figures preceding, such action tends to increase the "out-curve." But in the second it tends to decrease the "in-curve." The "in-curve" would not be possible if this action were the only cause of curvature. It ought to be less than the out-curve, so far as this action is effective, in the figures given.

If we have rotation in the opposite direction from that in the figure, or if the line representative of ω_z is positive instead of negative, the rolling of the ball, if any, upon a cushion of compressed air in front of it would act to decrease the "out-curve" and increase the "in-curve."

Relative Motion of a Body.—When a body at any instant has two simultaneous motions we can consider the body itself as having one of these motions and the *space occupied by the body* as having the other. The first motion is then that which the body would appear to have to an observer in space moving with space and unaware of his own motion. We call it therefore the relative motion of the body with reference to moving space.

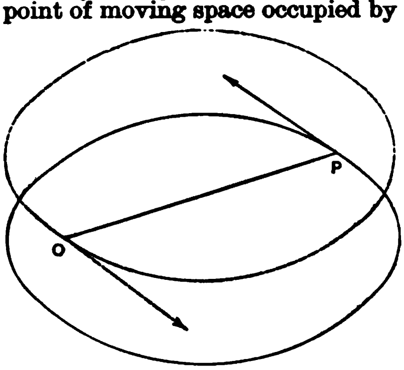
We have thus far seen how to determine the actual motion when we have given the relative motion and the motion of space. We have now to consider the inverse problem of how to determine the relative motion when we have given the actual motion and the motion of space.

We can solve the problem in two ways. We can resolve the given actual motion into two component motions one of which coincides with the given motion of space. Then the other must be the relative motion required. Or we can add to the actual motion, composed of these two component motions, a third motion equal and opposite to the given motion of space. This will counteract one of the components and leave as a result the relative motion.

Relative Motion of a Body with Reference to Space Translation.—If the actual motion of a body and the motion of the space occupied by it are both motions of translation, the relative motion will be one of translation also. In such case we can treat the body and the space occupied by it as points, and thus have simply to find the relative motion of a particle with reference to the point of moving space occupied by the particle.

The relative velocity of the particle is then the resultant of the actual velocity of the particle and the velocity of the point of moving space occupied by the particle, *taken as acting with reversed direction.*

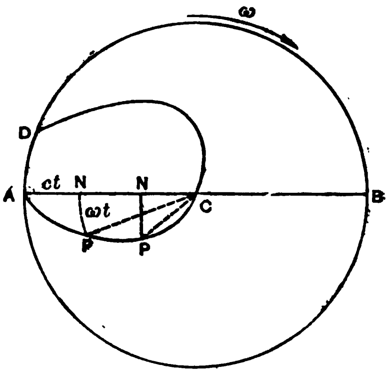
If then the actual velocity of the particle is zero, the relative velocity at any instant is always equal and opposite to that of the point of moving space occupied by the particle at that instant.



Thus if the particle P describes an ellipse with reference to the fixed point O at one focus, the relative velocity of O with reference to P will be always equal and opposite to the velocity of P at any instant, and the apparent path of O as seen from P will be a similar ellipse with P at a focus.

Relative Motion of a Body with Reference to Moving Space in General.—Any motion of a body at any instant can be resolved into a translation of any point, and a rotation about an axis through that point (page 190). If we take for this point the point of space occupied by any particle of the body, we have translation only of this point and particle, and the relative velocity is found as in the preceding Article.

The relative velocity of the particle is then, as before, the resultant of the actual velocity of the particle and the velocity of the point of moving space occupied by the particle, *taken as acting with reversed direction.*



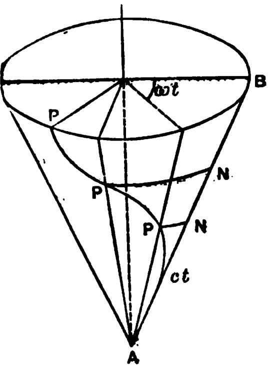
We obtain then the relative path by giving to the actual path the reversed motion of space.

For example, let the actual velocity of a particle P be uniform and equal to c, and its constant direction be in the direction AB. Let the line AB be the diameter of a circular disk which rotates clockwise with constant angular velocity ω about the axis at C. We obtain the path relative to the disk by supposing rotation of the actual path AB counter-clockwise.

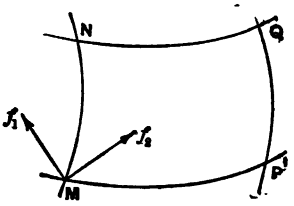
Thus at the end of any time t the particle has traversed the distance $AN = ct$ and the disk has turned clockwise through the angle ωt. The corresponding relative position P of the particle is then the point which N would occupy if the line AB were turned counter-clockwise through the angle ωt. Repeating the construction for successive values of t we obtain the relative path $APCD$. The end D corresponds to the rotation angle ωT, where T is the time of the actual motion from A to B. If $\omega T = \pi$, the point D would coincide with A. Let r be the radius of the disk. Then from the two equations $cT = 2r$, and $\omega T = \pi$, we have for the condition of this coincidence of D and A,

$$\frac{\omega}{c} = \frac{\pi}{2r}.$$

If the actual path AB makes an acute angle with the axis of rotation through C, the relative path lies on the surface of a cone.



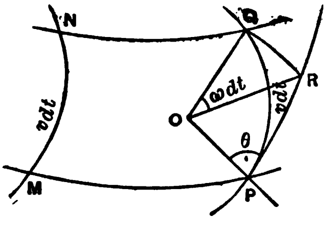
Acceleration of Relative Motion.—Let a particle describe the path MN with any motion, and at the same time let this path have a motion of translation. Then we can regard the first motion as relative with reference to the second, and its acceleration f_1 is the relative acceleration. Besides this relative acceleration at any instant, the particle has the acceleration f_2 of the motion of translation at that instant. The actual acceleration of the particle is then the resultant of the two accelerations f_1 and f_2.



It is, however, different when the path MN has any motion in general, because such motion may be resolved into a motion of translation of any point of the path and a motion of rotation about an axis through that point.

Take for this point the point of space occupied by the particle at any instant.

Then we have besides the acceleration f_1 of the particle in its path, and the acceleration f_2 of the point of space occupied by the particle, a third acceleration, f_3, due to the rotation, which we can determine as follows:



Let v be the relative velocity of the particle. Then in an indefinitely small time dt, vdt will be the element MN of the relative path. This element in the time dt is translated to PQ and at the same time has the angular velocity ω about the point of space occupied by P.

Let the axis OP through this point make the angle θ with PQ. Then at the end of the time dt, Q will be at R. If f_3 is the acceleration in the direction QR, then the distance QR will be

$$QR = \frac{1}{2} f_3 dt^2.$$

The radius of rotation is $OQ = vdt \sin \theta$. Hence the distance QR is also given by

$$QR = OQ \, . \, \omega dt = vdt \sin \theta \, . \, \omega dt.$$

Equating these two values of QR, we obtain

$$f_3 = 2v\omega \sin \theta.$$

Hence we see that the actual acceleration of the particle in general is the resultant of three accelerations:

The first, f_1, is the acceleration of the relative motion of the particle.

The second, f_2, is the acceleration of the point of space occupied by the particle.

The third, f_3, is equal in magnitude to twice the product of the relative velocity v of the particle, the angular velocity ω of the point of space occupied by the particle, and the sine of the angle θ which the element of the relative path makes with the axis through the point of space occupied by the particle. Its direction is at right angles to the plane of this axis and element, and it acts in the direction given by the rotation.

If then f_1, f_2, f_3, Fig. 1, represent these accelerations, we have by completing the polygon in Fig. 2 the actual acceleration f.

Fig. 1.

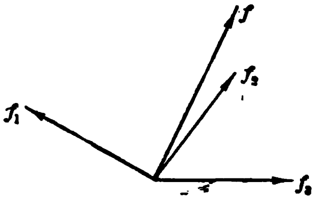
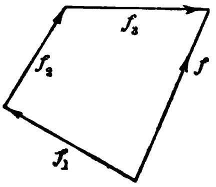
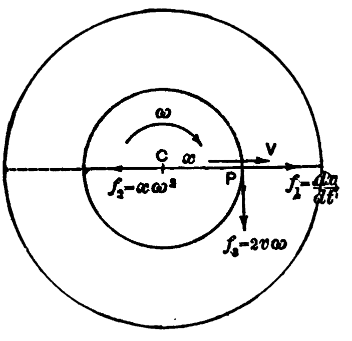


Fig. 2.



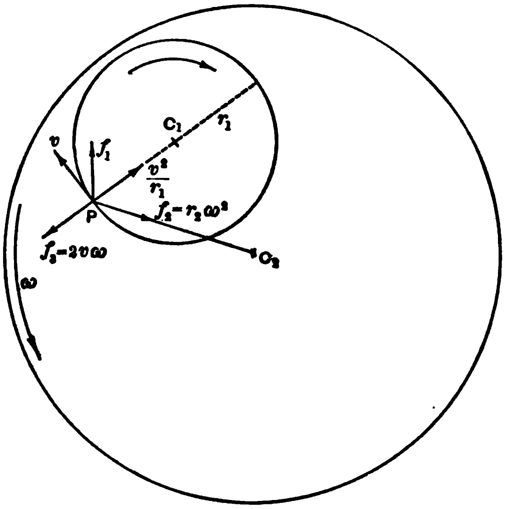
Inversely, if f, f_2 and f_3 are given and it is required to find the relative acceleration f_1, we must take f_2 and f_3 *reversed in direction.*

Illustrations.—Let a particle P move at any instant with the velocity v in the direction of a diameter of a circular disk. Let the disk at this instant have an angular velocity ω about its axis at C, and the distance CP of the particle from the axis be x.



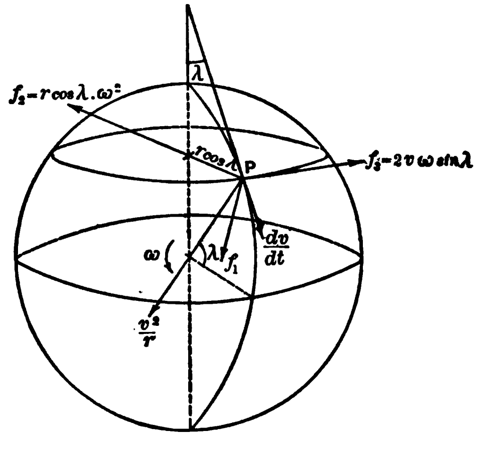
Then the relative acceleration is $f_1 = \frac{dv}{dt}$ along the diameter. The acceleration of the point of space occupied by the particle is $f_2 = x\omega^2$, along the diameter. This is the central acceleration of the point P due to rotation about C. We have also $f_3 = 2v\omega$, acting at right angles to the plane of the element of the relative path and the axis through P parallel to the axis at C, and it acts in the direction given by the rotation as shown. The angle θ which the element of the path makes with the parallel axis at P is 90°, and hence $\sin \theta = 1$. The actual acceleration is the resultant of these three accelerations.

Let a particle P move at any instant with the velocity v in the circumference of a circle of radius r_1 about the centre C_1, and at the same time let the centre C_1 revolve about the point C_2 with the angular velocity ω. Let the distance of P from C_2 be r_2.



Then the relative acceleration f_1 is the resultant of the tangential acceleration $\frac{dv}{dt}$ and the central acceleration $\frac{v^2}{r_1}$ acting towards C_1.

The acceleration f_2 of the point of space occupied by the particle is $r_2\omega^2$ acting towards C_2. We have also $f_3 = 2v\omega$ acting at right angles to the plane of the element of the relative path and the axis through P parallel to the axis at C_2, and it acts in the direction



given by the rotation as shown. The actual acceleration is the resultant of these three accelerations.

Let a particle P move on a meridian of the earth and have at any instant the velocity v and the tangential acceleration $\frac{dv}{dt}$. If r is the radius of the earth, the central acceleration is $\frac{v^2}{r}$ and the relative acceleration f_1 is the resultant of $\frac{dv}{dt}$ and $\frac{v^2}{r}$.

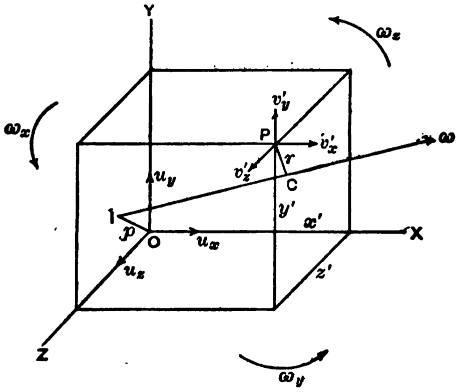
If λ is the latitude of P, $r\cos\lambda$ is the radius of rotation; and if ω is the angular velocity of the earth, the acceleration f_2 of the point of space occupied by the particle is $f_2 = r\cos\lambda \, . \, \omega^2$.

We have also $f_3 = 2v\omega \sin\lambda$ acting at right angles to the plane of the element of the relative path and the axis through P parallel to the earth's axis, that is, tangent to the latitude circle at P. It acts towards the east. The actual acceleration is the resultant of these three accelerations.

CHAPTER III.

GENERAL ANALYTICAL RELATIONS FOR A POINT OF A RIGID ROTATING SYSTEM. EULER'S GEOMETRIC EQUATIONS.

General Analytical Relations for a Point of a Rigid Rotating System.—Let a rigid system rotate at any instant about the axis IC with the angular velocity ω and the angular acceleration α. Take any point O of the system as origin, and let the direction-cosines of ω be $\cos\alpha$, $\cos\beta$, $\cos\gamma$.



Then we have for the components of ω and α

$$\left.\begin{array}{l}\omega_x = \omega\cos\alpha,\quad \omega_y = \omega\cos\beta,\quad \omega_z = \omega\cos\gamma;\\ \alpha_x = \alpha\cos\alpha,\quad \alpha_y = \alpha\cos\beta,\quad \alpha_z = \alpha\cos\gamma;\\ \cos\alpha = \dfrac{\omega_x}{\omega} = \dfrac{\alpha_x}{\alpha},\quad \cos\beta = \dfrac{\omega_y}{\omega} = \dfrac{\alpha_y}{\alpha},\quad \cos\gamma = \dfrac{\omega_z}{\omega} = \dfrac{\alpha_z}{\alpha};\end{array}\right\} \quad . \quad (1)$$

and since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$,

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2},\quad \alpha = \sqrt{\alpha_x^2 + \alpha_y^2 + \alpha_z^2}. \quad . \quad . \quad . \quad (2)$$

Let (x, y, z) be the co-ordinates of any point on the axis IC, and (x', y', z') the co-ordinates of any point P whose distance from the axis is $PC = r$. Then, as we have seen (page 190), we can resolve the rotation about IC into an equal angular velocity about a parallel axis through the origin O and a velocity $v = \omega p$ of the origin O, where p is the distance IO of the origin from the axis.

The components of this velocity of O are as on page 179, equation (1):

$$\left.\begin{aligned} v_x &= \omega_z y - \omega_y z; \\ v_y &= \omega_x z - \omega_z x; \\ v_z &= \omega_y x - \omega_x y. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The components of the linear velocity v' of P due to rotation about the parallel axis through the origin O are as on page 198, equations (6):

$$\left.\begin{aligned} v'_x &= \omega_y z' - \omega_z y'; \\ v'_y &= \omega_z x' - \omega_x z'; \\ v'_z &= \omega_x y' - \omega_y x'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have thus the total components of the velocity P, just as on page 198, equation (7):

$$\left.\begin{aligned} V_x &= v_x + v'_x; \\ V_y &= v_y + v'_y; \\ V_z &= v_z + v'_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The components of the linear tangential acceleration of P due to rotation about the parallel axis through O, we see from (4), are given by

$$\left.\begin{aligned} f'_{tx} &= \alpha_y z' - \alpha_z y'; \\ f'_{ty} &= \alpha_z x' - \alpha_x z'; \\ f'_{tz} &= \alpha_x y' - \alpha_y x'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since $p\omega = v$ and $p^2\omega^2 = v^2 = v_x^2 + v_y^2 + v_z^2$, we have from (3)

$$p^2 = \frac{(\omega_z y - \omega_y z)^2}{\omega^2} + \frac{(\omega_x z - \omega_z x)^2}{\omega^2} + \frac{(\omega_y x - \omega_x y)^2}{\omega^2}. \quad . \quad . \quad (7)$$

Let f'_n be the normal linear acceleration of the point P due to rotation about the parallel axis through the origin O. Then $f'_n = v'\omega$; and since velocity in the hodograph is the normal acceleration in the path (page 52), we have directly from the figure, for the components of f'_n,

$$\left.\begin{aligned} f'_{nx} &= v'_z\omega_y - v'_y\omega_z; \\ f'_{ny} &= v'_x\omega_z - v'_z\omega_x; \\ f'_{nz} &= v'_y\omega_x - v'_x\omega_y. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We have then for the components of the acceleration f' of the point P, from (8) and (6),

$$\left.\begin{aligned} f'_x &= f'_{nx} + f'_{tx} = (v'_z\omega_y - v'_y\omega_z) + (\alpha_y z' - \alpha_z y'); \\ f'_y &= f'_{ny} + f'_{ty} = (v'_x\omega_z - v'_z\omega_x) + (\alpha_z x - \alpha_x z'); \\ f'_z &= f'_{nz} + f'_{tz} = (v'_y\omega_x - v'_x\omega_y) + (\alpha_x y' - \alpha_y z'). \end{aligned}\right\} \quad . \quad . \quad (9)$$

If we put for v'_x, v'_y, v'_z their values as given in (4), we have

$$\left.\begin{aligned} f'_x &= (\omega_x x' + \omega_y y' + \omega_z z')\omega_x - \omega^2 x' + (\alpha_y z' - \alpha_z y'); \\ f'_y &= (\omega_x x' + \omega_y y' + \omega_z z')\omega_y - \omega^2 y' + (\alpha_z x' - \alpha_x z'); \\ f'_z &= (\omega_x x' + \omega_y y' + \omega_z z')\omega_z - \omega^2 z' + (\alpha_x y' - \alpha_y x'). \end{aligned}\right\} \quad . \quad . \quad (10)$$

Equations (10) give the values of the components of the linear acceleration f' of any point P of the system whose co-ordinates are (x', y', z'), in terms of these co-ordinates and the components of ω and α.

The moments of the component linear accelerations with reference to the origin O are

$$f'_x \sqrt{y'^2 + z'^2}, \quad f'_y \sqrt{x'^2 + z'^2}, \quad f'_z \sqrt{x'^2 + y'^2}. \quad . \quad . \quad . \quad (11)$$

For the moments about the axes of the components of v' and f' we have:

$$\left.\begin{array}{llll} \text{about } X \text{ parallel to plane} & YZ, & M_x = v'_z y' - v'_y z', \text{ or } & f'_z y' - f'_y z'; \\ \text{“} \quad Y \quad \text{“} \quad \text{“} \quad \text{“} & ZX, & M_y = v'_x z' - v'_z x', \text{ or } & f'_x z' - f'_z x'; \\ \text{“} \quad Z \quad \text{“} \quad \text{“} \quad \text{“} & XY, & M_z = v'_y x' - v'_x y', \text{ or } & f'_y x' - f'_x y'. \end{array}\right\} (12)$$

The resultant moment in both cases is given by

$$M_r = \sqrt{M_x^2 + M_y^2 + M_z^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Its line representative has the direction-cosines

$$\frac{M_x}{M_r}, \quad \frac{M_y}{M_r}, \quad \frac{M_z}{M_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Looking along the line representative towards the origin the rotation is counter-clockwise.

[We can deduce equations (9) directly by the Calculus. Thus if we differentiate the values of v'_x, v'_y, v'_z given by (4), then, since

$$\frac{dx'}{dt} = v'_x, \quad \frac{dy'}{dt} = v'_y, \quad \frac{dz'}{dt} = v'_z,$$

and

$$\frac{d\omega_x}{dt} = \alpha_x, \quad \frac{d\omega_y}{dt} = \alpha_y, \quad \frac{d\omega_z}{dt} = \alpha_z,$$

we have at once

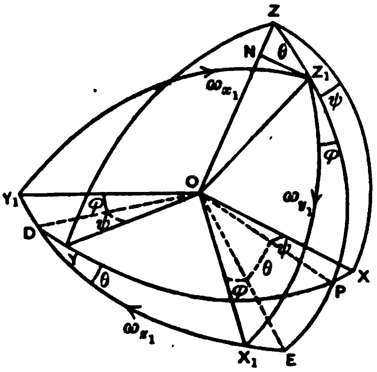
$$f'_x = \frac{dv'_x}{dt} = (v'_z\omega_y - v'_y\omega_z) + (\alpha_y z' - \alpha_z y');$$

$$f'_y = \frac{dv'_y}{dt} = (v'_x\omega_z - v'_z\omega_x) + (\alpha_z x' - \alpha_x z');$$

$$f'_z = \frac{dv'_z}{dt} = (v'_y\omega_x - v'_x\omega_y) + (\alpha_x y' - \alpha_y x').$$

Euler's Geometrical Equations.—*To determine the geometrical equations between the motion of a rigid system in space and the angular velocity of the system about an axis in the system.*

Let OX_1, OY_1, OZ_1 be rectangular co-ordinate axes, fixed in the system and therefore rotating with it, and let the system rotate about some axis fixed in the system, and therefore making invariable angles with these axes, so that the component angular velocities in the co-ordinate planes are ω_{x_1}, ω_{y_1}, ω_{z_1}. We take direction of rotation as always



$$\left.\begin{array}{l}\text{about } X \text{ from } Z \text{ to } Y\\ \text{“} \quad Y \quad \text{“} \quad X \text{ “ } Z\\ \text{“} \quad Z \quad \text{“} \quad Y \text{ “ } X\end{array}\right\}\begin{array}{l}\text{positive, the opposite}\\ \text{direction negative.}\end{array}$$

Let now OX, OY, OZ be rectangular co-ordinate axes whose directions *in space* are invariable. For instance, the axis OZ may be always directed towards the North Pole, then XY is the plane of the celestial equator.

Let the point O be taken as the centre of a sphere of radius r. Let X, Y, Z and X_1, Y_1, Z_1 be the points in which this sphere is pierced by the fixed and moving axes.

Let the axes OX_1, OY_1, OZ_1 have the initial positions OX, OY, OZ. First turn the system about OZ as an axis through the angle $XZP = \psi$, so that OX moves to OP, and OY to OD. Then turn the system about OD as an axis through the angle $ZOZ_1 = \theta$, so that OP moves to OE, and OZ to OZ_1. Finally turn the system about OZ_1 as an axis through the angle $EZX_1 = \phi$, so that OE moves to OX_1, and OD to OY_1.

It is required to find the geometric relations between θ, ϕ, ψ and ω_{x_1}, ω_{y_1}, ω_{z_1} as the system rotates. These geometric relations are called **Euler's Geometric Equations.**

Let the angular velocity of Z_1 perpendicular to the plane ZOZ_1 at any instant be denoted by $\frac{d\psi}{dt}$. This is called the **angular velocity of precession.** Let the angular velocity of Z_1 along ZZ_1 at the same instant be denoted by $\frac{d\theta}{dt}$. This is called the **angular velocity of nutation.** Let the angular velocity of X_1 with reference to E at that instant be denoted by $\frac{d\phi}{dt}$.

Draw Z_1N perpendicular to OZ. Then $Z_1N = r\sin\theta$, and the linear velocity at any instant of Z_1 perpendicular to the plane ZOZ_1 is $r\sin\theta \, . \, \frac{d\psi}{dt}$, and along ZZ_1 at the same instant it is $r\frac{d\theta}{dt}$. The linear velocity at the same instant of Z_1 along Y_1Z_1 is $r\omega_{x_1}$, and along Z_1X_1 it is $r\omega_{y_1}$.

We have then directly from the figure

$$r\frac{d\theta}{dt} = r\omega_{y_1}\cos\phi + r\omega_{x_1}\sin\phi;$$

$$r\sin\theta \, . \, \frac{d\psi}{dt} = r\omega_{y_1}\sin\phi - r\omega_{x_1}\cos\phi.$$

Since the radius r cancels out,

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_{y_1}\cos\phi + \omega_{x_1}\sin\phi; \\ \sin\theta\frac{d\psi}{dt} &= \omega_{y_1}\sin\phi - \omega_{x_1}\cos\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Combining these two equations,

$$\left.\begin{aligned} \omega_{x_1} &= \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi; \\ \omega_{y_1} &= \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the same way by drawing a perpendicular from E to $OZOE$ we have the linear velocity of E perpendicular to ZOE equal to $r\cos\theta\frac{d\psi}{dt}$, and of X_1 relative to E along EX_1, $r\frac{d\phi}{dt}$.

The whole velocity of X_1 in space along X_1Y_1 is $r\omega_{z_1}$. Hence

$$\omega_{z_1} = \frac{d\psi}{dt}\cos\theta + \frac{d\phi}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (1), (2) and (3) are Euler's Geometric Equations.

EXAMPLES.

(1) *Deduce the angular velocities* ω_x, ω_y, ω_z *about the fixed axis, in terms of* θ, ϕ, ψ.

Ans. Let ω_r be the resultant angular velocity about the fixed axes. If we impress on space and also on the system, in addition to its existing motion, an angular velocity equal to $-\omega_r$ about the resultant axis of rotation, the axes OX_1, OY_1, OZ_1 will become fixed and OX, OY, OZ will move with angular velocities $-\omega_x$, $-\omega_y$, $-\omega_z$. Hence in the equations already found we have only to replace ϕ by $-\psi$, θ by $-\theta$, ψ by $-\phi$, and ω_{x_1}, ω_{y_1}, ω_{z_1} will become $-\omega_x$, $-\omega_y$, $-\omega_z$, and we have

$$\left.\begin{aligned} \omega_x &= -\frac{d\theta}{dt}\sin\psi + \frac{d\phi}{dt}\sin\theta\cos\psi; \\ \omega_y &= \frac{d\theta}{dt}\cos\psi + \frac{d\phi}{dt}\sin\theta\sin\psi; \\ \omega_z &= \frac{d\phi}{dt}\cos\theta + \frac{d\psi}{dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(2) ***Refer the axes fixed in space to the axes fixed in the system.***

Ans. We have simply to interchange in the figure X_1, Y_1, Z_1 with X, Y, Z, each with each. If then the angles θ, ϕ, ψ are still measured as indicated in the figure, the relations connecting them with the angular velocities are obtained by changing ω_{x_1}, ω_{y_1}, ω_{z_1} into $-\omega_x$, $-\omega_y$, $-\omega_z$.

If we measure θ in the direction opposite to that indicated in the figure, the expressions for ω_x, ω_y are identical with those already found for ω_{x_1}, ω_{y_1}.

(3) *If p, q, r, are the direction-cosines of OZ with regard to the axes OX_1, OY_1, OZ_1, show that*

$$\left.\begin{aligned} \frac{dp}{dt} - q\omega_{z_1} + r\omega_{y_1} &= 0; \\ \frac{dq}{dt} - r\omega_{x_1} + p\omega_{z_1} &= 0; \\ \frac{dr}{dt} - p\omega_{y_1} + q\omega_{x_1} &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5)$$

Ans. Any one of these may be obtained by differentiating one of the expressions

$$p = -\sin\theta\cos\phi, \quad q = \sin\theta\sin\phi, \quad r = \cos\theta$$

(4) *Show that the direction-cosines of either set of Euler's axes with regard to the other are given by*

$$\left.\begin{aligned} \cos XX_1 &= -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta; \\ \cos YX_1 &= \cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta; \\ \cos ZX_1 &= -\sin\theta\cos\phi. \end{aligned}\right\} \quad \ldots \quad (6)$$

$$\left.\begin{aligned} \cos XY_1 &= -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta; \\ \cos YY_1 &= \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta; \\ \cos ZY_1 &= \sin\theta\sin\phi. \end{aligned}\right\} \quad \ldots \quad (7)$$

$$\left.\begin{aligned} \cos XZ_1 &= \sin\theta\cos\psi; \\ \cos YZ_1 &= \sin\theta\sin\psi; \\ \cos ZZ_1 &= \cos\theta. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (8)$$

Ans. We have from the figure the following spherical triangles for which we know two sides and the included angle :

Triangle.	Sides.	Angle.	Triangle.	Sides.	Angle.
DX_1X	$DX = 90^\circ + \psi$ $DX_1 = 90 - \phi$	$XDX_1 = \theta$	DXY_1	$Y_1D = \phi$ $DX = 90 + \psi$	$Y_1DX = 180 - \theta$
DYX_1	$DY = \psi$ $DX_1 = 90 - \phi$	$YDX_1 = \theta$	DYY_1	$Y_1D = \phi$ $YD = \psi$	$Y_1DY = 180 - \theta$
Z_1ZX_1	$ZZ_1 = \theta$ $Z_1X_1 = 90$	$ZZ_1X_1 = 180 - \phi$	Z_1ZY_1	$Y_1Z_1 = 90$ $Z_1Z = \theta$	$Y_1Z_1Z = 90 - \phi$

Triangle.	Sides.	Angle.
PXZ_1	$PX = \psi$ $PZ_1 = 90 - \theta$	$Z_1PX_1 = 90^\circ$
PYZ_1	$PZ_1 = 90 - \theta$ $PY = 90 - \psi$	$YPZ_1 = 90$

Solving these triangles we have at once equations (6), (7), (8).

(5) *Prove in the same way the following:*

$$\left.\begin{aligned} \cos X_1X &= -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta; \\ \cos Y_1X &= -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta; \\ \cos Z_1X &= \sin\theta\cos\psi. \end{aligned}\right\} \quad \ldots \quad (9)$$

$$\left.\begin{aligned} \cos X_1Y &= \cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta; \\ \cos Y_1Y &= \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta; \\ \cos Z_1Y &= \sin\theta\sin\psi. \end{aligned}\right\} \quad \ldots \quad (10)$$

$$\left.\begin{aligned}\cos X_1Z &= -\sin\theta\cos\phi;\\ \cos Y_1Z &= \sin\theta\sin\phi;\\ \cos Z_1Z &= \cos\theta.\end{aligned}\right\} \quad \ldots\ldots \quad (11)$$

(6) ***Find the relations between the co-ordinates x, y, z, of the fixed system of axes and the co-ordinates x_1, y_1, z_1 of the moving system.***

Ans. If we multiply the first of equations (6) by x, the second by y, the third by z and add, and do the same for (7) and (8), we have at once, as we see from the figure,

$$\left.\begin{aligned}x_1 &= (-\sin\psi\sin\phi+\cos\psi\cos\phi\cos\theta)x\\ &\quad+(\cos\psi\sin\phi+\sin\psi\cos\phi\cos\theta)y-\sin\theta\cos\phi.z;\\ y_1 &= (-\sin\psi\cos\phi-\cos\psi\sin\phi\cos\theta)x\\ &\quad+(\cos\psi\cos\phi-\sin\psi\sin\phi\cos\theta)y+\sin\theta\sin\phi.z;\\ z_1 &= \sin\theta\cos\psi.x+\sin\theta\sin\psi.y+\cos\theta.z.\end{aligned}\right\} \quad \ldots \quad (12)$$

In the same way we have from equations (9), (10), (11),

$$\left.\begin{aligned}x &= (-\sin\psi\sin\phi+\cos\psi\cos\phi\cos\theta)x_1\\ &\quad+(-\sin\psi\cos\phi-\cos\psi\sin\phi\cos\theta)y_1+\sin\theta\cos\psi.z_1;\\ y &= (\cos\psi\sin\phi+\sin\psi\cos\phi\cos\theta)x_1\\ &\quad+(\cos\psi\cos\phi-\sin\psi\sin\phi\cos\theta)y_1+\sin\theta\sin\psi.z_1;\\ z &= -\sin\theta\cos\phi.x_1+\sin\theta\sin\phi.y_1+\cos\theta.z_1.\end{aligned}\right\} \quad . \quad (13)$$

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